

# Volumes of orthogonal groups and unitary groups

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## Abstract

The matrix integral has many applications in diverse fields. This review article begins by presenting detailed key background knowledge about matrix integral. Then the volumes of orthogonal groups and unitary groups are computed, respectively. Applications are also presented as well. Specifically, The volume of the set of mixed quantum states is computed by using the volume of unitary group. The volume of a metric ball in unitary group is also computed as well.

There are no new results in this article, but only detailed and elementary proofs of existing results. The purpose of the article is pedagogical, and to collect in one place many, if not all, of the quantum information applications of the volumes of orthogonal and unitary groups.

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# 1 Introduction

Volumes of orthogonal groups and unitary groups are very useful in physics and mathematics [2, 3]. In 1949, Ponting and Potter had already calculated the volume of orthogonal and unitary group [15]. A complete treatment for group manifolds is presented by Marinov [8], who extracted the volumes of groups by studying curved path integrals [9]. There is a general closed formula for any compact Lie group in terms of the root lattice [12]. Clearly the methods used previously are not followed easily. Życzkowski in his paper [25] gives a re-derivation on the volume of unitary group, but it is still not accessible. The main goal of this paper is to compute the volumes of orthogonal groups and unitary groups in a systematic and elementary way. The obtained formulae is more applicable.

As an application, by re-deriving the volumes of orthogonal groups and unitary groups, the authors in [25] computes the volume of the convex  $(n^2 - 1)$ -dimensional set  $D(\mathbb{C}^n)$  of the density matrices of size  $n$  with respect to the Hilbert-Schmidt measure. Recently, the authors in [17] give the integral representation of the exact volume of a metric ball in unitary group; and present diverse applications of volume estimates of metric balls in manifolds in information and coding theory.

Before proceeding, we recall some notions in group theory. Assume that a group  $\mathcal{G}$  acts on the underlying vector space  $\mathcal{X}$  via  $g|x\rangle$  for all  $g \in \mathcal{G}$  and  $x \in \mathcal{X}$ . Let  $|x\rangle$  be any nonzero vector in  $\mathcal{X}$ . The subset  $\mathcal{G}|x\rangle := \{g|x\rangle : g \in \mathcal{G}\}$  is called the  $\mathcal{G}$ -orbit of  $|x\rangle \in \mathcal{X}$ . Denote

$$x^{\mathcal{G}} := \{g \in \mathcal{G} : g|x\rangle = |x\rangle\}.$$

We have the following fact:

$$\mathcal{G}|x\rangle \sim \mathcal{G}/x^{\mathcal{G}}.$$

If  $x^{\mathcal{G}} = \{e\}$ , where  $e$  is a unit element of  $\mathcal{G}$ , then the action of  $\mathcal{G}$  on  $|x\rangle$  is called *free*. In this case,

$$\mathcal{G}|x\rangle \sim \mathcal{G}/\{e\} \iff \mathcal{G} \sim \mathcal{G}|x\rangle.$$

Now we review a fast track of the volume of a unitary group. Let us consider the unitary groups  $\mathcal{G} = \mathcal{U}(n+1), n = 1, 2, \dots$ . To establish their structure, we look at spaces in which the group acts transitively, and identify the isotropy subgroup. The unitary group  $\mathcal{U}(n+1)$  acts naturally in the complex vector space  $\mathcal{X} = \mathbb{C}^{n+1}$  through the vector or "defining" representation; the image of any nonzero vector  $|x\rangle = |\psi\rangle \in \mathbb{C}^{n+1}$  is contained in the maximal sphere  $\mathbb{S}^{2n+1}$  of radius  $\|\psi\|$  since  $\|\psi\| = \|U\psi\|$  for  $U \in \mathcal{U}(n+1)$ , and in fact it is easy to see that it sweeps the whole sphere when  $U$  runs through  $\mathcal{U}(n+1)$ , i.e. the group  $\mathcal{U}(n+1)$  acts transitively in this sphere. The isotropy group of the vector  $|n+1\rangle$  is easily seen to be the unitary group with an entry less, that is  $x^{\mathcal{G}} = \mathcal{U}(n) \oplus 1$ . Indeed, the following map is *onto*:

$$\varphi : \mathcal{U}(n+1) \longrightarrow \mathcal{U}(n+1)|n+1\rangle = \mathbb{S}^{2n+1}.$$

If we identify  $\mathcal{U}(n)$  with  $\mathcal{U}(n) \oplus 1$ , as a subgroup of  $\mathcal{U}(n+1)$ , then

$$\mathcal{U}(n)|n+1\rangle = |n+1\rangle,$$

implying that  $\ker \varphi = \mathcal{U}(n)$ , and thus

$$\mathcal{U}(n+1)/\ker \varphi \sim \mathcal{U}(n+1)|n+1\rangle = \mathbb{S}^{2n+1} = (\mathcal{U}(n+1)/\ker \varphi)|n+1\rangle.$$

Therefore we have the equivalence relation

$$\mathcal{U}(n+1)/\mathcal{U}(n) = \mathbb{S}^{2n+1}. \quad (1.1)$$

This indicates that

$$\text{vol}(\mathcal{U}(n+1)) = \text{vol}(\mathbb{S}^{2n+1}) \cdot \text{vol}(\mathcal{U}(n)), \quad n = 1, 2, \dots \quad (1.2)$$

That is,

$$\text{vol}(\mathcal{U}(n)) = \text{vol}(\mathbb{S}^1) \times \text{vol}(\mathbb{S}^3) \times \dots \times \text{vol}(\mathbb{S}^{2n-1}). \quad (1.3)$$

We can see this in [5]. The volume of the sphere of unit radius embedded in  $\mathbb{R}^n (n \geq 1)$ , is calculated from the Gaussian integral:

$$\sqrt{\pi} = \int_{-\infty}^{+\infty} e^{-t^2} dt.$$

Now

$$\begin{aligned} (\sqrt{\pi})^n &= \left( \int_{-\infty}^{+\infty} e^{-t^2} dt \right)^n = \int_{\mathbb{R}^n} e^{-\|v\|^2} dv \\ &= \int_0^{+\infty} \int_{\mathbb{S}^{n-1}(r)} e^{-r^2} d\sigma dr = \int_0^{+\infty} \sigma_{n-1}(r) e^{-r^2} dr, \end{aligned}$$

where  $\sigma_{n-1}(r) = \int_{\mathbb{S}^{n-1}(r)} d\sigma$  is the volume of sphere  $\mathbb{S}^{n-1}(r)$  of radius  $r$ . Since

$$\sigma_{n-1}(r) = \sigma_{n-1}(1) \times r^{n-1},$$

it follows that

$$(\sqrt{\pi})^n = \sigma_{n-1}(1) \times \int_0^{+\infty} r^{n-1} e^{-r^2} dr,$$

implying that

$$\text{vol}(\mathbb{S}^{n-1}) := \frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}. \quad (1.4)$$

Finally, we get the volume formula of a unitary group:

$$\text{vol}(\mathcal{U}(n)) := \prod_{k=1}^n \frac{2\pi^k}{\Gamma(k)} = \frac{2^n \pi^{\frac{n(n+1)}{2}}}{1!2! \cdots (n-1)!}. \quad (1.5)$$

## 2 Volumes of orthogonal groups

### 2.1 Preliminary

The following standard notations will be used [10]. Scalars will be denoted by lower-case letters, vectors and matrices by capital letters. As far as possible variable matrices will be denoted by  $X, Y, \dots$  and constant matrices by  $A, B, \dots$

Let  $A = [a_{ij}]$  be a  $n \times n$  matrix, then  $\text{Tr}(A) = \sum_{j=1}^n a_{jj}$  is the trace of  $A$ , and  $\det(A)$  is the determinant of  $A$ , and  $^\top$  over a vector or a matrix will denote its transpose. Let  $X = [x_{ij}]$  be a

$m \times n$  matrix of independent real entries  $x_{ij}$ 's. We denote the matrix of differentials by  $dX$ , i.e.

$$dX := [dx_{ij}] = \begin{bmatrix} dx_{11} & dx_{12} & \cdots & dx_{1n} \\ dx_{21} & dx_{22} & \cdots & dx_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ dx_{m1} & dx_{m2} & \cdots & dx_{mn} \end{bmatrix}.$$

Then  $[dX]$  stands for the product of the  $m \times n$  differential elements

$$[dX] := \prod_{i=1}^m \prod_{j=1}^n dx_{ij} \quad (2.1)$$

and when  $X$  is a real square symmetric matrix, that is,  $m = n$ ,  $X = X^T$ , then  $[dX]$  is the product of the  $n(n+1)/2$  differential elements, that is,

$$[dX] := \prod_{j=1}^n \prod_{i=j}^n dx_{ij} = \prod_{i \geq j} dx_{ij}. \quad (2.2)$$

Throughout this paper, stated otherwise, we will make use of conventions that the signs will be ignored in the product  $[dX]$  of differentials of independent entries. Our notation will be the following: Let  $X = [x_{ij}]$  be a  $m \times n$  matrix of independent real entries. Then

$$\boxed{[dX] = \wedge_{i=1}^m \wedge_{j=1}^n dx_{ij}} \quad (2.3)$$

when  $[dX]$  appears with integrals or Jacobians of transformations;

$$\boxed{[dX] = \prod_{i=1}^m \prod_{j=1}^n dx_{ij}} \quad (2.4)$$

when  $[dX]$  appears with integrals involving *density functions* where [the functions are nonnegative and the absolute value of the Jacobian is automatically taken](#).

It is assumed that the reader is familiar with the calculation of Jacobians when a vector of scalar variables is transformed to a vector of scalar variables. The result is stated here for the sake of completeness. Let the vector of scalar variables  $X$  be transformed to  $Y$ , where

$$X = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \quad \text{and} \quad Y = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix},$$

by a one-to-one transformation. Let the matrix of partial derivatives be denoted by

$$\frac{\partial Y}{\partial X} = \left[ \frac{\partial y_i}{\partial x_j} \right].$$

The determinant of the matrix  $\left[\frac{\partial y_i}{\partial x_j}\right]$  is known as the *Jacobian* of the transformation  $X$  going to  $Y$  or  $Y$  as a function of  $X$  it is written as

$$J(Y : X) = \det \left( \left[ \frac{\partial y_i}{\partial x_j} \right] \right) \quad \text{or} \quad [dY] = J(Y : X)[dX], \quad J \neq 0$$

and

$$J(Y : X) = \frac{1}{J(X : Y)} \quad \text{or} \quad 1 = J(Y : X)J(X : Y).$$

Note that when transforming  $X$  to  $Y$  the variables can be taken in any order because a permutation brings only a change of sign in the determinant and the magnitude remains the same, that is,  $|J|$  remains the same where  $|J|$  denotes the absolute value of  $J$ . When evaluating integrals involving functions of matrix arguments one often needs only the absolute value of  $J$ . Hence in all the statements of this notes the notation  $[dY] = J[dX]$  means that the relation is written ignoring the sign.

**Proposition 2.1.** Let  $X, Y \in \mathbb{R}^n$  be of independent real variables and  $A \in \mathbb{R}^{n \times n}$  be a nonsingular matrix of constants. If  $Y = AX$ , then

$$\boxed{[dY] = \det(A)[dX].} \tag{2.5}$$

*Proof.* The result follows from the definition itself. Note that when  $Y = AX$ ,  $A = [a_{ij}]$  one has

$$y_i = a_{i1}x_1 + \cdots + a_{in}x_n, \quad i = 1, \dots, n$$

where  $x_j$ 's and  $y_j$ 's denote the components of the vectors  $X$  and  $Y$ , respectively. Thus the partial derivative of  $y_i$  with respect to  $x_j$  is  $a_{ij}$ , and then the determinant of the Jacobian matrix is  $\det(A)$ .  $\square$

In order to see the results in the more complicated cases we need the concept of a *tensor product*.

**Definition 2.2** (Tensor product). Let  $A = [a_{ij}] \in \mathbb{R}^{p \times q}$  and  $B = [b_{ij}] \in \mathbb{R}^{m \times n}$ . Then the *tensor product*, denoted by  $\otimes$ , is a  $pm \times qn$  matrix in  $\mathbb{R}^{pm \times qn}$ , formed as follows:

$$B \otimes A = \begin{bmatrix} a_{11}B & a_{12}B & \cdots & a_{1q}B \\ a_{21}B & a_{22}B & \cdots & a_{2q}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{p1}B & a_{p2}B & \cdots & a_{pq}B \end{bmatrix} \tag{2.6}$$

and

$$A \otimes B = \begin{bmatrix} b_{11}A & b_{12}A & \cdots & b_{1n}A \\ b_{21}A & b_{22}A & \cdots & b_{2n}A \\ \vdots & \vdots & \ddots & \vdots \\ b_{m1}A & b_{m2}A & \cdots & b_{mn}A \end{bmatrix}. \quad (2.7)$$

**Definition 2.3** (Vector-matrix correspondence). Let  $X = [x_{ij}] \in \mathbb{R}^{m \times n}$  matrix. Let the  $j$ -th column of  $X$  be denoted by  $X_j$ . That is,  $X = [X_1, \dots, X_n]$ , where

$$X_j = \begin{bmatrix} x_{1j} \\ \vdots \\ x_{mj} \end{bmatrix}.$$

Consider an  $mn$ -dimensional vector in  $\mathbb{R}^{mn}$ , formed by appending  $X_1, \dots, X_n$  and forming a long string. This vector will be denoted by  $\text{vec}(X)$ . That is,

$$\text{vec}(X) = \begin{bmatrix} X_1 \\ \vdots \\ X_n \end{bmatrix}. \quad (2.8)$$

From the above definition, we see that the  $\text{vec}$  mapping is a one-to-one and onto correspondence from  $\mathbb{R}^{m \times n}$  to  $\mathbb{R}^n \otimes \mathbb{R}^m$ . We also see that  $\text{vec}(AXB) = (A \otimes B^T) \text{vec}(X)$  if the product  $AXB$  exists.

**Proposition 2.4.** Let  $X, Y \in \mathbb{R}^{m \times n}$  be of independent real variables and  $A \in \mathbb{R}^{m \times m}$  and  $B \in \mathbb{R}^{n \times n}$  nonsingular matrices of constants. If  $Y = AXB$ , then

$$\boxed{[dY] = \det(A)^n \det(B)^m [dX]}. \quad (2.9)$$

*Proof.* Since  $Y = AXB$ , it follows that  $\text{vec}(Y) = (A \otimes B^T) \text{vec}(X)$ . Then by using Proposition 2.1, we have

$$\begin{aligned} J(Y : X) &= \det\left(\frac{\partial Y}{\partial X}\right) := \det\left(\frac{\partial(\text{vec}(Y))}{\partial(\text{vec}(X))}\right) \\ &= \det(A \otimes B^T) = \det(A \otimes \mathbb{1}_n) \det(\mathbb{1}_m \otimes B^T) \\ &= \det(A)^n \det(B)^m, \end{aligned}$$

implying that

$$[dY] = J(Y : X)[dX] = \det(A)^n \det(B)^m [dX].$$

This completes the proof. □

**Remark 2.5.** Another approach to the proof that  $[dZ] = \det(A)^n [dX]$ , where  $Z = AX$ , is described as follows: we partition  $Z$  and  $X$ , respectively, as:  $Z = [Z_1, \dots, Z_n]$ ,  $X = [X_1, \dots, X_n]$ . Now  $Z = AX$  can be rewritten as  $Z_j = AX_j$  for all  $j$ . So

$$\frac{\partial Z}{\partial X} = \frac{\partial(Z_1, \dots, Z_n)}{\partial(X_1, \dots, X_n)} = \begin{bmatrix} A & & & \\ & A & & \\ & & \ddots & \\ & & & A \end{bmatrix}, \quad (2.10)$$

implying that

$$[dZ] = \det \left( \begin{bmatrix} A & & & \\ & A & & \\ & & \ddots & \\ & & & A \end{bmatrix} \right) [dX] = \det(A)^n [dX].$$

**Proposition 2.6.** Let  $X, A, B \in \mathbb{R}^{n \times n}$  be lower triangular matrices where  $A = [a_{ij}]$  and  $B = [b_{ij}]$  are constant matrices with  $a_{jj} > 0, b_{jj} > 0, j = 1, \dots, n$  and  $X$  is a matrix of independent real variables. Then

$$Y = X + X^T \implies [dY] = 2^n [dX], \quad (2.11)$$

$$Y = AX \implies [dY] = \left( \prod_{j=1}^n a_{jj}^j \right) [dX], \quad (2.12)$$

$$Y = XB \implies [dY] = \left( \prod_{j=1}^n b_{jj}^{n-j+1} \right) [dX]. \quad (2.13)$$

Thus

$$Y = AXB \implies [dY] = \left( \prod_{j=1}^n a_{jj}^j b_{jj}^{n-j+1} \right) [dX]. \quad (2.14)$$

*Proof.*  $Y = X + X^T$  implies that

$$\begin{bmatrix} x_{11} & 0 & \cdots & 0 \\ x_{21} & x_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{nn} \end{bmatrix} + \begin{bmatrix} x_{11} & x_{21} & \cdots & x_{n1} \\ 0 & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & x_{nn} \end{bmatrix} = \begin{bmatrix} 2x_{11} & x_{21} & \cdots & x_{n1} \\ x_{21} & 2x_{22} & \cdots & x_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1} & x_{n2} & \cdots & 2x_{nn} \end{bmatrix}.$$

When taking the partial derivatives the  $n$  diagonal elements give 2 each and others unities and hence  $[dY] = 2^n [dX]$ . If  $Y = AX$ , then the matrices of the configurations of the partial derivatives, by taking the elements in the orders  $(y_{11}, y_{21}, \dots, y_{n1}); (y_{22}, \dots, y_{n2}); \dots; y_{nn}$  and

$(x_{11}, x_{21}, \dots, x_{n1}); (x_{22}, \dots, x_{n2}); \dots; x_{nn}$  are the following:

$$\begin{aligned} \frac{\partial(y_{11}, y_{21}, \dots, y_{n1})}{\partial(x_{11}, x_{21}, \dots, x_{n1})} &= A, \quad \frac{\partial(y_{22}, \dots, y_{n2})}{\partial(x_{22}, \dots, x_{n2})} = A[\hat{1}|\hat{1}], \dots, \\ \frac{\partial y_{nn}}{\partial x_{nn}} &= A[\hat{1} \cdots \widehat{n-1} | \hat{1} \cdots \widehat{n-1}] = a_{nn}, \end{aligned}$$

where  $A[\hat{i}_1 \cdots \hat{i}_\mu | \hat{j}_1 \cdots \hat{j}_\nu]$  means that the obtained submatrix from deleting both the  $i_1, \dots, i_\mu$ -th rows and the  $j_1, \dots, j_\nu$ -th columns of  $A$ . Thus

$$\begin{aligned} \frac{\partial Y}{\partial X} &= \begin{bmatrix} \frac{\partial(y_{11}, y_{21}, \dots, y_{n1})}{\partial(x_{11}, x_{21}, \dots, x_{n1})} & 0 & \cdots & 0 \\ 0 & \frac{\partial(y_{22}, \dots, y_{n2})}{\partial(x_{22}, \dots, x_{n2})} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{\partial y_{nn}}{\partial x_{nn}} \end{bmatrix} \\ &= \begin{bmatrix} A & 0 & \cdots & 0 \\ 0 & A[\hat{1}|\hat{1}] & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A[\hat{1} \cdots \widehat{n-1} | \hat{1} \cdots \widehat{n-1}] \end{bmatrix}. \end{aligned}$$

We can also take another approach to this proof. In fact, we partition  $X, Y$  by columns, respectively,  $Y = [Y_1, \dots, Y_n]$  and  $X = [X_1, \dots, X_n]$ . Then  $Y = AX$  is equivalent to  $Y_j = AX_j, j = 1, \dots, n$ . Since  $Y, X, A$  are lower triangular, it follows that

$$\begin{bmatrix} y_{11} \\ y_{21} \\ \vdots \\ y_{n1} \end{bmatrix} = A \begin{bmatrix} x_{11} \\ x_{21} \\ \vdots \\ x_{n1} \end{bmatrix}, \quad \begin{bmatrix} y_{22} \\ \vdots \\ y_{n2} \end{bmatrix} = A[\hat{1}|\hat{1}] \begin{bmatrix} x_{22} \\ \vdots \\ x_{n2} \end{bmatrix}, \dots, y_{nn} = A[\hat{1} \cdots \widehat{n-1} | \hat{1} \cdots \widehat{n-1}] x_{nn} = a_{nn} x_{nn}.$$

Now

$$\begin{aligned} [dY] &= \prod_{j=1}^n [dY_j] = \det(A) \det(A[\hat{1}|\hat{1}]) \cdots \det(A[\hat{1} \cdots \widehat{n-1} | \hat{1} \cdots \widehat{n-1}]) \prod_{j=1}^n [dX_j] \\ &= \left( \prod_{j=1}^n a_{jj}^j \right) [dX]. \end{aligned}$$

Next if  $Y = XB$ , that is,

$$\begin{aligned}
 Y = XB &= \begin{bmatrix} x_{11} & 0 & \cdots & 0 \\ x_{21} & x_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{nn} \end{bmatrix} \begin{bmatrix} b_{11} & 0 & \cdots & 0 \\ b_{21} & b_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{bmatrix} \\
 &= \begin{bmatrix} x_{11}b_{11} & 0 & \cdots & 0 \\ x_{21}b_{11} + x_{22}b_{21} & x_{22}b_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{j=1}^n x_{nj}b_{j1} & \sum_{j=1}^n x_{nj}b_{j2} & \cdots & x_{nn}b_{nn} \end{bmatrix}
 \end{aligned}$$

The matrices of the configurations of the partial derivatives, by taking the elements in the orders  $y_{11}; (y_{21}, y_{22}); \dots; (y_{n1}, \dots, y_{nn})$  and  $x_{11}; (x_{21}, x_{22}); \dots; (x_{n1}, \dots, x_{nn})$  are the following:

$$\begin{aligned}
 \frac{\partial y_{11}}{\partial x_{11}} &= b_{11}, \quad \frac{\partial(y_{21}, y_{22})}{\partial(x_{21}, x_{22})} = \begin{bmatrix} b_{11} & b_{21} \\ 0 & b_{22} \end{bmatrix} = \begin{bmatrix} b_{11} & 0 \\ b_{21} & b_{22} \end{bmatrix}^T, \\
 \frac{\partial(y_{31}, y_{32}, y_{33})}{\partial(x_{31}, x_{32}, x_{33})} &= \begin{bmatrix} b_{11} & b_{21} & b_{31} \\ 0 & b_{22} & b_{32} \\ 0 & 0 & b_{33} \end{bmatrix} = \begin{bmatrix} b_{11} & 0 & 0 \\ b_{21} & b_{22} & 0 \\ b_{31} & b_{32} & b_{33} \end{bmatrix}^T, \dots, \\
 \frac{\partial(y_{n1}, y_{n2}, \dots, y_{nn})}{\partial(x_{n1}, x_{n2}, \dots, x_{nn})} &= \begin{bmatrix} b_{11} & 0 & \cdots & 0 \\ b_{21} & b_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{bmatrix}^T.
 \end{aligned}$$

Thus

$$\frac{\partial Y}{\partial X} = \begin{bmatrix} \frac{\partial y_{11}}{\partial x_{11}} & 0 & \cdots & 0 \\ 0 & \frac{\partial(y_{21}, y_{22})}{\partial(x_{21}, x_{22})} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{\partial(y_{n1}, y_{n2}, \dots, y_{nn})}{\partial(x_{n1}, x_{n2}, \dots, x_{nn})} \end{bmatrix}$$

Denote by  $B[i_1 \dots i_\mu | j_1 \dots j_\nu]$  the sub-matrix formed by the  $i_1, \dots, i_\mu$ -th rows and  $j_1, \dots, j_\nu$ -th columns of  $B$ . Hence

$$\frac{\partial Y}{\partial X} = \begin{bmatrix} B[1|1] & 0 & \cdots & 0 \\ 0 & B[12|12] & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & B[1 \dots n | 1 \dots n] \end{bmatrix}^T.$$

The whole configuration is a upper triangular matrix with  $b_{11}$  appearing  $n$  times and  $b_{22}$  appearing  $n - 1$  times and so on in the diagonal. Also we give another approach to derive the Jacobian for  $Y = XB$ . Indeed, we partition  $Y, X$  by rows, respectively,

$$Y = \begin{bmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_n \end{bmatrix}, \quad X = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{bmatrix},$$

where  $Y_j, X_j$  are row-vectors. So  $Y = XB$  is equivalent to  $Y_j = X_j B, j = 1, \dots, n$ . Moreover

$$\begin{aligned} y_{11} &= x_{11}B[1|1] = x_{11}a_{11}, [y_{21}, y_{22}] = [x_{21}, x_{22}]B[12|12], \dots, \\ [y_{n1}, \dots, y_{nn}] &= [y_{n1}, \dots, y_{nn}]B[1 \dots n|1 \dots n]. \end{aligned}$$

Therefore

$$\begin{aligned} [dY] &= \prod_{j=1}^n [dY_j] = \det(B[1|1]) \det(B[12|12]) \cdots \det(B[1 \dots n|1 \dots n]) \prod_{j=1}^n [dX_j] \\ &= \left( \prod_{j=1}^n b_{jj}^{n+j-1} \right) [dX]. \end{aligned}$$

We are done. □

**Proposition 2.7.** *Let  $X$  be a lower triangular matrix of independent real variables and  $A = [a_{ij}]$  and  $B = [b_{ij}]$  be lower triangular matrices of constants with  $a_{jj} > 0, b_{jj} > 0, j = 1, \dots, n$ . Then*

$$Y = AX + X^T A^T \implies [dY] = 2^n \left( \prod_{j=1}^n a_{jj}^j \right) [dX], \quad (2.15)$$

$$Y = XB + B^T X^T \implies [dY] = 2^n \left( \prod_{j=1}^n b_{jj}^{n-j+1} \right) [dX]. \quad (2.16)$$

*Proof.* Let  $Z = AX$ . Then  $Y = Z + Z^T$ . Thus  $[dY] = 2^n [dZ]$ . Since  $Z = AX$ , it follows from Proposition 2.6 that

$$[dZ] = \left( \prod_{j=1}^n a_{jj}^j \right) [dX],$$

implying the result. The proof of the second identity goes similarly. □

**Proposition 2.8.** *Let  $X$  and  $Y$  be  $n \times n$  symmetric matrices of independent real variables and  $A \in \mathbb{R}^{n \times n}$  nonsingular matrix of constants. If  $Y = AXA^T$ , then*

$$\boxed{[dY] = \det(A)^{n+1} [dX]}. \quad (2.17)$$

*Proof.* Since both  $X$  and  $Y$  are symmetric matrices and  $A$  is nonsingular we can split  $A$  and  $A^\top$  as products of elementary matrices and write in the form

$$Y = \cdots E_2 E_1 X E_1^\top E_2^\top \cdots$$

where  $E_j, j = 1, 2, \dots$  are elementary matrices. Write  $Y = AXA^\top$  as a sequence of transformations of the type

$$Y_1 = E_1 X E_1^\top, Y_2 = E_2 Y_1 E_2^\top, \dots, \implies \\ [dY_1] = J(Y_1 : X)[dX], [dY_2] = J(Y_2 : Y_1)[dY_1], \dots$$

Now successive substitutions give the final result as long as the Jacobians of the type  $J(Y_k : Y_{k-1})$  are computed. Note that the elementary matrices are formed by multiplying any row (or column) of an identity matrix with a scalar, adding a row (column) to another row (column) and combinations of these operations. Hence we need to consider only these two basic elementary matrices. Let us consider a  $3 \times 3$  case and compute the Jacobians. Let  $E_1$  be the elementary matrix obtained by multiplying the first row by  $\alpha$  and  $E_2$  by adding the first row to the second row of an identity matrix. That is,

$$E_1 = \begin{bmatrix} \alpha & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and

$$E_1 X E_1^\top = \begin{bmatrix} \alpha^2 x_{11} & \alpha x_{12} & \alpha x_{13} \\ \alpha x_{21} & x_{22} & x_{23} \\ \alpha x_{31} & x_{32} & x_{33} \end{bmatrix}, \\ E_2 Y_1 E_2^\top = \begin{bmatrix} u_{11} & u_{11} + u_{12} & u_{13} \\ u_{11} + u_{21} & u_{11} + u_{21} + u_{12} + u_{22} & u_{13} + u_{23} \\ u_{31} & u_{31} + u_{32} & u_{33} \end{bmatrix},$$

where  $Y_1 = E_1 X E_1^\top$  and  $Y_2 = E_2 Y_1 E_2^\top$  and the elements of  $Y_1$  are denoted by  $u_{ij}$ 's for convenience. The matrix of partial derivatives in the transformation  $Y_1$  written as a function of  $X$  is then

$$\frac{\partial Y_1}{\partial X} = \begin{bmatrix} \alpha^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

This is obtained by taking the  $x_{ij}$ 's in the order  $x_{11}; (x_{21}, x_{22}); (x_{31}, x_{32}, x_{33})$  and the  $u_{ij}$ 's also in the same order. Thus the Jacobian is given by

$$J(Y_1 : X) = \alpha^4 = \alpha^{3+1} = \det(E_1)^{3+1}.$$

Or by definition it follows directly that

$$[dY_1] = d(\alpha^2 x_{11}) d(\alpha x_{12}) d(\alpha x_{13}) dx_{22} dx_{23} dx_{33} = \alpha^4 [dX].$$

For a  $n \times n$  matrix it will be  $\alpha^{n+1}$ . Let the elements of  $Y_2$  be denoted by  $v_{ij}$ 's. Then again taking the variables in the order as in the case of  $Y_1$  written as a function of  $X$  the matrix of partial derivatives in this transformation is the following:

$$\frac{\partial Y_2}{\partial Y_1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

The determinant of this matrix is  $1 = 1^{3+1} = \det(E_2)^{3+1}$ . In general such a transformation gives the Jacobian, in absolute value, as  $1 = 1^{n+1}$ . Thus the Jacobian is given by

$$J(Y : X) = \det(\cdots E_2 E_1)^{n+1} = \det(A)^{n+1}.$$

We are done. □

**Example 2.9.** Let  $X \in \mathbb{R}^{n \times n}$  be a real symmetric positive definite matrix having a matrix-variate gamma distribution with parameters  $(\alpha, B = B^\top > 0)$ . We show that

$$\det(B)^{-\alpha} = \frac{1}{\Gamma_n(\alpha)} \int_{X>0} [dX] \det(X)^{\alpha - \frac{n+1}{2}} e^{-\text{Tr}(BX)}, \quad \text{Re}(\alpha) > \frac{n-1}{2}. \quad (2.18)$$

Indeed, since  $B$  is symmetric positive definite there exists a nonsingular matrix  $C$  such that  $B = CC^\top$ . Note that

$$\text{Tr}(BX) = \text{Tr}(C^\top XC).$$

Let

$$U = C^\top XC \implies [dU] = \det(C)^{n+1} [dX]$$

from Proposition 2.8 and  $\det(X) = \det(B)^{-1} \det(U)$ . The integral on the right reduces to the following:

$$\int_{X>0} [dX] \det(X)^{\alpha - \frac{n+1}{2}} e^{-\text{Tr}(BX)} = \det(B)^{-\alpha} \int_{U>0} [dU] \det(U)^{\alpha - \frac{n+1}{2}} e^{-\text{Tr}(U)}.$$

But

$$\int_{U>0} [dU] \det(U)^{\alpha - \frac{n+1}{2}} e^{-\text{Tr}(U)} = \Gamma_n(\alpha)$$

for  $\text{Re}(\alpha) > \frac{n-1}{2}$ . The result is obtained.

**Proposition 2.10.** *Let  $X, Y \in \mathbb{R}^{n \times n}$  skew symmetric matrices of independent real variables and  $A \in \mathbb{R}^{n \times n}$  nonsingular matrix of constants. If  $Y = AXA^\top$ , then*

$$\boxed{[dY] = \det(A)^{n-1} [dX]}. \quad (2.19)$$

Note that when  $X$  is skew symmetric the diagonal elements are zeros and hence there are only  $\frac{n(n-1)}{2}$  independent variables in  $X$ .

**Proposition 2.11.** *Let  $X, A, B \in \mathbb{R}^{n \times n}$  be lower triangular matrices where  $A = [a_{ij}]$  and  $B = [b_{ij}]$  are nonsingular constant matrices with positive diagonal elements, respectively, and  $X$  is a matrix of independent real variables. Then*

$$Y = A^\top X + X^\top A \implies [dY] = 2^n \left( \prod_{j=1}^n a_{jj}^j \right) [dX], \quad (2.20)$$

$$Y = XB^\top + BX^\top \implies [dY] = 2^n \left( \prod_{j=1}^n b_{jj}^{n-j+1} \right) [dX]. \quad (2.21)$$

*Proof.* Consider  $Y = A^\top X + X^\top A$ . Premultiply by  $(A^\top)^{-1}$  and postmultiply by  $A^{-1}$  to get the following:

$$Y = A^\top X + X^\top A \implies (A^\top)^{-1} Y A^{-1} = (A^\top)^{-1} X^\top + X A^{-1}.$$

Let

$$Z = X A^{-1} + (X A^{-1})^\top \implies [dZ] = 2^n \left( \prod_{j=1}^n a_{jj}^{-(n-j+1)} \right) [dX]$$

by Proposition 2.7 and

$$Z = (A^{-1})^\top Y A^{-1} \implies [dZ] = \det(A)^{-(n+1)} [dY]$$

by Proposition 2.8. Now writing  $[dY]$  in terms of  $[dX]$  one has

$$[dY] = \left( \prod_{j=1}^n a_{jj}^{-(n-j+1)} \right) 2^n \left( \prod_{j=1}^n a_{jj}^{n+1} \right) [dX] = 2^n \left( \prod_{j=1}^n a_{jj}^j \right) [dX]$$

since  $\det(A) = \prod_{j=1}^n a_{jj}$  because  $A$  is lower triangular. Thus the first result follows. The second is proved as follows. Clearly,

$$Y = BX^\top + XB^\top \implies B^{-1}Y(B^\top)^{-1} = B^{-1}X + (B^{-1}X)^\top := Z.$$

Thus  $[dZ] = \det(B)^{-(n+1)}[dY]$  and  $[dZ] = 2^n \left( \prod_{j=1}^n a_{jj}^{-j} \right) [dX]$ . Therefore expressing  $[dY]$  in terms of  $[dX]$  gives the second result.  $\square$

**Proposition 2.12.** *Let  $X \in \mathbb{R}^{n \times n}$  be a symmetric positive definite matrix of independent real variables and  $T = [t_{ij}]$  a real lower triangular matrix with  $t_{jj} > 0, j = 1, \dots, n$ , and  $t_{ij}, i \geq j$  independent. Then*

$$X = T^\top T \implies [dX] = 2^n \left( \prod_{j=1}^n t_{jj}^j \right) [dT], \quad (2.22)$$

$$X = TT^\top \implies [dX] = 2^n \left( \prod_{j=1}^n t_{jj}^{n-j+1} \right) [dT]. \quad (2.23)$$

*Proof.* By considering the matrix of differentials one has

$$X = TT^\top \implies dX = dT \cdot T^\top + T \cdot dT^\top.$$

Now treat this as a linear transformation in the differentials, that is,  $dX$  and  $dT$  as variables and  $T$  a constant. This completes the proof.  $\square$

**Example 2.13.** Let  $X \in \mathbb{R}^{n \times n}$  symmetric positive definite matrix and  $\operatorname{Re}(\alpha) > \frac{n-1}{2}$ . Show that

$$\begin{aligned} \Gamma_n(\alpha) &:= \int_{X>0} [dX] \det(X)^{\alpha - \frac{n+1}{2}} e^{-\operatorname{Tr}(X)} \\ &= \pi^{\frac{n(n-1)}{4}} \Gamma(\alpha) \Gamma\left(\alpha - \frac{1}{2}\right) \cdots \Gamma\left(\alpha - \frac{n-1}{2}\right). \end{aligned} \quad (2.24)$$

Let  $T$  be a real lower triangular matrix with positive diagonal elements. Then the unique representation

$$X = TT^\top \implies [dX] = 2^n \left( \prod_{j=1}^n t_{jj}^{n-j+1} \right) [dT].$$

Note that

$$\begin{aligned}\text{Tr}(X) &= \text{Tr}(TT^\top) = t_{11}^2 + (t_{21}^2 + t_{22}^2) + \cdots + (t_{n1}^2 + \cdots + t_{nn}^2), \\ \det(X) &= \det(TT^\top) = \prod_{j=1}^n t_{jj}^2.\end{aligned}$$

When  $X > 0$ , we have  $TT^\top > 0$ , but  $t_{jj} > 0, j = 1, \dots, n$  which means that  $-\infty < t_{ij} < \infty, i > j, 0 < t_{jj} < \infty, j = 1, \dots, n$ . The integral splits into  $n$  integrals on  $t_{jj}$ 's and  $\frac{n(n-1)}{2}$  integrals on  $t_{ij}$ 's,  $i > j$ . That is,

$$\Gamma_n(\alpha) = \left( \prod_{j=1}^n 2 \int_0^\infty (t_{jj}^2)^{\alpha - \frac{n+1}{2}} t_{jj}^{n-j+1} e^{-t_{jj}^2} dt_{jj} \right) \times \left( \prod_{i>j} \int_{-\infty}^\infty e^{-t_{ij}^2} dt_{ij} \right).$$

But

$$2 \int_0^\infty (t_{jj}^2)^{\alpha - \frac{j}{2}} e^{-t_{jj}^2} dt_{jj} = \Gamma\left(\alpha - \frac{j-1}{2}\right),$$

for  $\text{Re}(\alpha) > \frac{j-1}{2}, j = 1, \dots, n$  and

$$\int_{-\infty}^\infty e^{-t_{ij}^2} dt_{ij} = \sqrt{\pi}.$$

Multiplying them together the result follows. Note that

$$\text{Re}(\alpha) > \frac{j-1}{2}, j = 1, \dots, n \implies \text{Re}(\alpha) > \frac{n-1}{2}.$$

## 2.2 The computation of volumes

**Definition 2.14** (Stiefel manifold). Let  $A$  be a  $n \times m$  ( $n \geq m$ ) matrix with real entries such that  $A^\top A = \mathbb{1}_m$ , that is, the  $m$  columns of  $A$  are *orthonormal vectors*. The set of all such matrices  $A$  is known as the *Stiefel manifold*, denoted by  $\mathcal{O}(m, n)$ . That is, for all  $n \times m$  matrices  $A$ ,

$$\boxed{\mathcal{O}(m, n) = \{A \in \mathbb{R}^{n \times m} : A^\top A = \mathbb{1}_m\}}, \quad (2.25)$$

where  $\mathbb{R}^{n \times m}$  denotes the set of all  $n \times m$  real matrices.

The equation  $A^\top A = \mathbb{1}_m$  imposes  $m(m+1)/2$  conditions on the elements of  $A$ . Thus the number of independent entries in  $A$  is  $mn - m(m+1)/2$ .

If  $m = n$ , then  $A$  is an orthogonal matrix. The set of such orthogonal matrices form a group. This group is known as the *orthogonal group* of  $m \times m$  matrices.

**Definition 2.15** (Orthogonal group). Let  $B$  be a  $n \times n$  matrix with real elements such that  $B^\top B = \mathbb{1}_n$ . The set of all  $B$  is called an *orthogonal group*, denoted by  $\mathcal{O}(n)$ . That is,

$$\boxed{\mathcal{O}(n) = \{B \in \mathbb{R}^{n \times n} : B^\top B = \mathbb{1}_n\}}. \quad (2.26)$$

Clearly  $\mathcal{O}(n, n) = \mathcal{O}(n)$ . Note that  $B^\top B = \mathbb{1}_n$  imposes  $n(n+1)/2$  conditions and hence the number of independent entries in  $B$  is only  $n^2 - n(n+1)/2 = n(n-1)/2$ .

**Definition 2.16** (A symmetric or a skew symmetric matrix). Let  $A \in \mathbb{R}^{n \times n}$ . If  $A = A^\top$ , then  $A$  is said to be *symmetric* and if  $A^\top = -A$ , then it is *skew symmetric*.

**Proposition 2.17.** Let  $V \in \mathcal{O}(n)$  with independent entries and the diagonal entries or the entries in the first row of  $V$  all positive. Denote  $dG = V^\top dV$  where  $V = [v_1, \dots, v_n]$ . Then

$$[dG] = \prod_{i=1}^{n-1} \prod_{j=i+1}^n \langle v_i, dv_j \rangle \quad (2.27)$$

$$= 2^{n(n-1)/2} \det(\mathbb{1}_n + X)^{-(n-1)} [dX], \quad (2.28)$$

where  $X$  is a skew symmetric matrix such that the first row entries of  $(\mathbb{1}_n + X)^{-1}$ , except the first entry, are negative.

*Proof.* Let the columns of  $V$  be denoted by  $v_1, \dots, v_n$ . Since the columns are orthonormal, we have  $\langle v_i, v_j \rangle = \delta_{ij}$ . Then

$$\langle v_i, dv_j \rangle + \langle dv_i, v_j \rangle = 0,$$

implying  $\langle v_j, dv_i \rangle = 0$  since  $\langle v_j, dv_i \rangle$  is a real scalar. We also have

$$\langle v_i, dv_j \rangle = -\langle v_j, dv_i \rangle \text{ for } i \neq j.$$

Then  $V^\top dV$  is a skew symmetric matrix. That is,

$$\begin{aligned} dG &= V^\top dV = \begin{bmatrix} v_1^\top \\ v_2^\top \\ \vdots \\ v_n^\top \end{bmatrix} [dv_1, dv_2, \dots, dv_n] \\ &= \begin{bmatrix} \langle v_1, dv_1 \rangle & \langle v_1, dv_2 \rangle & \cdots & \langle v_1, dv_n \rangle \\ \langle v_2, dv_1 \rangle & \langle v_2, dv_2 \rangle & \cdots & \langle v_2, dv_n \rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle v_n, dv_1 \rangle & \langle v_n, dv_2 \rangle & \cdots & \langle v_n, dv_n \rangle \end{bmatrix}. \end{aligned}$$

This indicates that

$$dG = \begin{bmatrix} 0 & \langle v_1, dv_2 \rangle & \cdots & \langle v_1, dv_n \rangle \\ -\langle v_1, dv_2 \rangle & 0 & \cdots & \langle v_2, dv_n \rangle \\ \vdots & \vdots & \ddots & \vdots \\ -\langle v_1, dv_n \rangle & -\langle v_2, dv_n \rangle & \cdots & 0 \end{bmatrix}$$

Then there are only  $n(n-1)/2$  independent entries in  $G$ . Then  $[dG]$  is the wedge product of the entries upper the leading diagonal in the matrix  $V^T dV$ :

$$[dG] = \wedge_{i=1}^{n-1} \wedge_{j=i+1}^n \langle v_i, dv_j \rangle$$

This establishes (2.27). For establishing (2.28), take a skew symmetric matrix  $X$ , then  $V = 2(\mathbb{1}_n + X)^{-1} - \mathbb{1}_n$  is orthonormal such that  $VV^T = \mathbb{1}_n$ . Further the matrix of differentials in  $V$  is given by  $dV = -2(\mathbb{1}_n + X)^{-1} \cdot dX \cdot (\mathbb{1}_n + X)^{-1}$ , i.e.

$$dV = -\frac{1}{2}(\mathbb{1}_n + V) \cdot dX \cdot (\mathbb{1}_n + V).$$

Thus

$$dG = V^T dV = -\frac{1}{2}(\mathbb{1}_n + V^T) \cdot dX \cdot (\mathbb{1}_n + V)$$

and the wedge product is obtained

$$[dG] = \det \left( \frac{\mathbb{1}_n + V^T}{\sqrt{2}} \right)^{n-1} [dX] = \det \left( \sqrt{2}(\mathbb{1}_n + X)^{-1} \right)^{n-1} [dX].$$

Therefore the desired identity (2.28) is proved.  $\square$

**Proposition 2.18.** *Let  $X$  be a  $n \times n$  symmetric matrix of independent real entries and with distinct and nonzero eigenvalues  $\lambda_1 > \cdots > \lambda_n$  and let  $D = \text{diag}(\lambda_1, \dots, \lambda_n)$ . Let  $V \in \mathcal{O}(n)$  be a unique such that  $X = VDV^T$ . Then*

$$\boxed{[dX] = \left( \prod_{i=1}^{n-1} \prod_{j=i+1}^n |\lambda_i - \lambda_j| \right) [dD][dG]}, \quad (2.29)$$

where  $dG = V^T dV$ .

*Proof.* Take the differentials in  $X = VDV^T$  to get

$$dX = dV \cdot D \cdot V^T + V \cdot dD \cdot V^T + V \cdot D \cdot dV^T,$$

implying that

$$V^T \cdot dX \cdot V = V^T dV \cdot D + dD + D \cdot dV^T V.$$

Let  $dY = V^\top \cdot dX \cdot V$  for fixed  $V$ . Since  $dG = V^\top dV$ , it follows that  $[dX] = [dY]$  and

$$\begin{aligned} dY &= dG \cdot D + dD + D \cdot dG^\top \\ &= dD + dG \cdot D - D \cdot dG \\ &= dD + [dG, D] \end{aligned}$$

where we used the fact that  $dG^\top = -dG$  and the concept of commutator, defined by  $[M, N] := MN - NM$ . Clearly the commutator of  $M$  and  $N$  is skew symmetric. Now

$$dy_{jj} = d\lambda_j \text{ and } dy_{ij} = (\lambda_j - \lambda_i)dg_{ij}, \quad i < j = 1, \dots, n.$$

Then

$$\begin{aligned} [dY] &= \left( \prod_{j=1}^n dy_{jj} \right) \left( \prod_{i < j} dy_{ij} \right) = \left( \prod_{j=1}^n d\lambda_j \right) \left( \prod_{i < j} |\lambda_i - \lambda_j| dg_{ij} \right) \\ &= \left( \prod_{i < j} |\lambda_i - \lambda_j| \right) [dD][dG], \end{aligned}$$

where  $[dD] = \prod_{j=1}^n d\lambda_j$  and  $[dG] = \prod_{i < j} dg_{ij}$ , and the desired conclusion is obtained.  $\square$

Clearly  $X = VDV^\top$ , where  $D = \text{diag}(\lambda_1, \dots, \lambda_n)$  ( $\lambda_1 > \dots > \lambda_n$ ),  $VV^\top = \mathbb{1}_n$ , is not a one-to-one transformation from  $X$  to  $(D, V)$  since  $X$  determines  $2^n$  matrices  $[\pm v_1, \dots, \pm v_n]$ , where  $v_1, \dots, v_n$  are the columns of  $V$ , such that  $X = VDV^\top$ .

This transformation can be shown to be unique if one entry from each row and column are of a specified sign, for example, the diagonal entries are positive. Once this is done we are integrating with respect to  $dG$ , where  $dG = V^\top dV$  over the full orthogonal group  $\mathcal{O}(n)$ , the result must be divided by  $2^n$  to get the result for a unique transformation  $X = VDV^\top$ .

**Remark 2.19.** Now we can try to compute the following integral based on Eq. (2.29):

$$\begin{aligned} \int_{X > 0: \text{Tr}(X)=1} [dX] &= \int_{\lambda_1 > \dots > \lambda_n > 0} \delta \left( \sum_{j=1}^n \lambda_j - 1 \right) \prod_{i < j} |\lambda_i - \lambda_j| \prod_{j=1}^n d\lambda_j \times \int_{\mathcal{O}(n)} [dG] \\ &= \frac{1}{n!} \int_0^\infty \delta \left( \sum_{j=1}^n \lambda_j - 1 \right) \prod_{i < j} |\lambda_i - \lambda_j| \prod_{j=1}^n d\lambda_j \times \frac{1}{2^n} \int_{\mathcal{O}(n)} [dG] \\ &= \frac{1}{2^n n!} \frac{1}{C_n^{(1,1)}} \text{vol}(\mathcal{O}(n)), \end{aligned}$$

that is,

$$\text{vol}(D(\mathbb{R}^n)) := \int_{X > 0: \text{Tr}(X)=1} [dX] = \frac{\pi^{\frac{n(n-1)}{4}} \Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n(n+1)}{4}\right) \Gamma\left(\frac{1}{2}\right)} \prod_{j=1}^n \Gamma\left(\frac{j}{2}\right).$$

See below for the notation  $C_n^{(1,1)}$  and  $\text{vol}(\mathcal{O}(n))$ .

**Proposition 2.20.** Let  $X$  be a  $p \times n$  ( $p \leq n$ ) matrix of rank  $p$  and let  $X = TU_1^T$ , where  $T$  is a  $p \times p$  lower triangular matrix with distinct nonzero diagonal entries and  $U_1$  is a unique  $n \times p$  semi-orthogonal matrix,  $U_1^T U_1 = \mathbb{1}_p$ , all are of independent real entries. Let  $U_2$  be an  $n \times (n - p)$  semi-orthogonal matrix such that  $U_1$  augmented with  $U_2$  is a full orthogonal matrix. That is,  $U = [U_1 \ U_2]$ ,  $U^T U = \mathbb{1}_n$ ,  $U_2^T U_2 = \mathbb{1}_{n-p}$ ,  $U_1^T U_2 = 0$ . Let  $u_j$  be the  $j$ -th column of  $U$  and  $du_j$  its differential. Then

$$[dX] = \left( \prod_{j=1}^p |t_{jj}|^{n-j} \right) [dT][dU_1], \quad (2.30)$$

where

$$[dU_1] = \wedge_{j=1}^p \wedge_{i=j+1}^n \langle u_i, du_j \rangle.$$

*Proof.* Note that

$$U^T U = \begin{bmatrix} U_1^T \\ U_2^T \end{bmatrix} [U_1 \ U_2] = \begin{bmatrix} U_2^T U_2 & U_1^T U_2 \\ U_2^T U_1 & U_1^T U_1 \end{bmatrix} = \begin{bmatrix} \mathbb{1}_p & 0 \\ 0 & \mathbb{1}_{n-p} \end{bmatrix}.$$

Take the differentials in  $X = TU_1^T$  to get

$$dX = dT \cdot U_1^T + T \cdot dU_1^T.$$

Then

$$\begin{aligned} dX \cdot U &= dT \cdot U_1^T U + T \cdot dU_1^T U \\ &= dT \cdot U_1^T [U_1, \ U_2] + T \cdot dU_1^T [U_1, \ U_2] \\ &= [dT + T \cdot dU_1^T \cdot U_1, \ T \cdot dU_1^T \cdot U_2] \end{aligned}$$

since  $U_1^T U_1 = \mathbb{1}_p$ ,  $U_1^T U_2 = 0$ . Make the substitutions

$$dW = dX \cdot U, dY = dU_1^T \cdot U_1, dS = dU_1^T \cdot U_2, dH = T \cdot dS.$$

Now we have

$$dW = [dT + T \cdot dY, \ dH].$$

Thus

$$dT + T \cdot dY = \begin{bmatrix} dt_{11} & 0 & \cdots & 0 \\ dt_{21} & dt_{22} & \cdots & \vdots \\ \vdots & \vdots & \ddots & 0 \\ dt_{p1} & dt_{p2} & \cdots & dt_{pp} \end{bmatrix} + \begin{bmatrix} t_{11} & 0 & \cdots & 0 \\ t_{21} & t_{22} & \cdots & \vdots \\ \vdots & \vdots & \ddots & 0 \\ t_{p1} & t_{p2} & \cdots & t_{pp} \end{bmatrix} \begin{bmatrix} 0 & dy_{12} & \cdots & dy_{1p} \\ -dy_{12} & 0 & \cdots & dy_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ -dy_{1p} & -dy_{2p} & \cdots & 0 \end{bmatrix}.$$

Let us consider, for example, the case where  $p = 2, 3$  in order for computing the wedge product of  $dT + T \cdot dY$ . Now for  $p = 2$ , we have

$$\begin{aligned} dT + T \cdot dY &= \begin{bmatrix} dt_{11} & 0 \\ dt_{21} & dt_{22} \end{bmatrix} + \begin{bmatrix} t_{11} & 0 \\ t_{21} & t_{22} \end{bmatrix} \begin{bmatrix} 0 & dy_{12} \\ -dy_{12} & 0 \end{bmatrix} \\ &= \begin{bmatrix} dt_{11} & t_{11}dy_{12} \\ dt_{21} - t_{22}dy_{12} & dt_{22} + t_{21}dy_{12} \end{bmatrix} \end{aligned}$$

Thus the wedge product of  $dT + T \cdot dY$  is:

$$\begin{aligned} [dT + T \cdot dY] &= dt_{11} \wedge (t_{11}dy_{12}) \wedge (dt_{21} - t_{22}dy_{12}) \wedge (dt_{22} + t_{21}dy_{12}) \\ &= t_{11}dt_{11} \wedge dy_{12} \wedge dt_{21} \wedge dt_{22} = t_{11}[dT][dY] \\ &= \left( \prod_{j=1}^2 |t_{jj}|^{2-j} \right) [dT][dY]. \end{aligned}$$

For  $p = 3$ , we have

$$\begin{aligned} dT + T \cdot dY &= \begin{bmatrix} dt_{11} & 0 & 0 \\ dt_{21} & dt_{22} & 0 \\ dt_{31} & dt_{32} & dt_{33} \end{bmatrix} + \begin{bmatrix} t_{11} & 0 & 0 \\ t_{21} & t_{22} & 0 \\ t_{31} & t_{32} & t_{33} \end{bmatrix} \begin{bmatrix} 0 & dy_{12} & dy_{13} \\ -dy_{12} & 0 & dy_{23} \\ -dy_{13} & -dy_{23} & 0 \end{bmatrix} \\ &= \begin{bmatrix} dt_{11} & t_{11}dy_{12} & t_{11}dy_{13} \\ dt_{21} - t_{22}dy_{12} & dt_{22} + t_{21}dy_{12} & t_{21}dy_{13} + t_{22}dy_{23} \\ dt_{31} - t_{32}dy_{12} - t_{33}dy_{13} & dt_{32} + t_{31}dy_{12} - t_{33}dy_{23} & dt_{33} + t_{31}dy_{13} + t_{32}dy_{23} \end{bmatrix}, \end{aligned}$$

implying the wedge product of  $dT + T \cdot dY$  is:

$$t_{11}^2 t_{22} [dT][dY] = \left( \prod_{j=1}^3 |t_{jj}|^{3-j} \right) [dT][dY].$$

For the general  $p$ , by straight multiplication, and remembering that the variables are only

$$dy_{12}, \dots, dy_{1p}, dy_{23}, \dots, dy_{2p}, \dots, dy_{p-1p}.$$

Thus the wedge product of  $dT + T \cdot dY$  gives

$$\left( \prod_{j=1}^p |t_{jj}|^{p-j} \right) [dY][dT],$$

ignoring the sign, and

$$[dY] = \wedge_{j=1}^{p-1} \wedge_{i=j+1}^p \langle u_i, du_j \rangle.$$

Now consider  $dH = T \cdot dS$ . Since  $dS$  is a  $p \times (n - p)$  matrix, we have

$$[dH] = \det(T)^{n-p} [dS] = \left( \prod_{j=1}^p |t_{jj}|^{n-p} \right) [dS].$$

The wedge product in  $dS$  is the following:

$$[dS] = \wedge_{j=1}^p \wedge_{i=p+j}^n \langle u_i, du_j \rangle.$$

Hence from the above equations,

$$\begin{aligned} [dX] &= [dW] = \wedge [dT + TdY] \wedge [dH] \\ &= \left( \prod_{j=1}^p |t_{jj}|^{p-j} \right) [dY][dT] \left( \prod_{j=1}^p |t_{jj}|^{n-p} \right) [dS] \\ &= \left( \prod_{j=1}^p |t_{jj}|^{n-j} \right) [dY][dT][dS]. \end{aligned}$$

Now

$$[dY][dS] = \wedge_{j=1}^{p-1} \wedge_{i=j+1}^p \langle u_i, du_j \rangle \wedge_{j=1}^p \wedge_{i=p+j}^n \langle u_i, du_j \rangle.$$

Substituting back one has

$$[dX] = \left( \prod_{j=1}^p |t_{jj}|^{n-j} \right) [dT] \wedge_{j=1}^p \wedge_{i=j+1}^n \langle u_i, du_j \rangle$$

which establishes the result.  $\square$

If the triangular matrix  $T$  is restricted to the one with positive diagonal entries, that is,  $t_{jj} > 0, j = 1, \dots, p$ , then while integrating over  $T$  using Proposition 2.20, the result must be multiplied by  $2^p$ . Without the factor  $2^p$ , the  $t_{jj}$ 's must be integrated over  $-\infty < t_{jj} < \infty, j = 1, \dots, p$ . If the expression to be integrated contains both  $T$  and  $U$ , then restrict  $t_{jj} > 0, j = 1, \dots, p$  and integrate  $U$  over the full Stiefel manifold. If the rows of  $U$  are  $u_1, \dots, u_p$ , then  $\pm u_1, \dots, \pm u_p$  give  $2^p$  choices. Similarly  $t_{jj} > 0, t_{jj} < 0$  give  $2^p$  choices. But there are not  $2^{2p}$  choices in  $X = TU$ . There are only  $2^p$  choices. Hence either integrate out the  $t_{jj}$ 's over  $-\infty < t_{jj} < \infty$  and a unique  $U$  or over  $0 < t_{jj} < \infty$  and the  $U$  over the full Stiefel manifold.

**Proposition 2.21.** *Let  $X_1$  be an  $n \times p$  ( $n \geq p$ ) matrix of rank  $p$  of independent real entries and let  $X_1 = U_1 T_1$  where  $T_1$  is a real  $p \times p$  upper triangular matrix with distinct nonzero diagonal entries and  $U_1$  is a unique real  $n \times p$  semi-orthogonal matrix, that is,  $U_1^T U_1 = \mathbb{1}_p$ . Let  $U = [U_1 \ U_2]$  such that  $U^T U = \mathbb{1}_n, U_2^T U_2 = \mathbb{1}_{n-p}, U_1^T U_2 = 0$ . Let  $u_j$  be the  $j$ -th column of  $U$  and  $du_j$  its differential. Then*

$$\boxed{[dX_1] = \left( \prod_{j=1}^p |t_{jj}|^{n-j} \right) [dT_1][dU_1],} \quad (2.31)$$

where

$$[dU_1] = \prod_{j=1}^p \prod_{i=j+1}^n \langle u_i, du_j \rangle.$$

**Proposition 2.22.** If  $X_1, T_1$  and  $U_1$  are as defined in Proposition 2.21, then the surface area of the full Stiefel manifold  $\mathcal{O}(p, n)$  or the total integral of the wedge product  $\wedge_{j=1}^p \wedge_{i=j+1}^n \langle u_i, \mathbf{d}u_j \rangle$  over  $\mathcal{O}(p, n)$  is given by

$$\int_{\mathcal{O}(p, n)} \wedge_{j=1}^p \wedge_{i=j+1}^n \langle u_i, \mathbf{d}u_j \rangle = \frac{2^p \pi^{\frac{pn}{2}}}{\Gamma_p\left(\frac{n}{2}\right)}, \quad (2.32)$$

where

$$\Gamma_p(\alpha) = \pi^{\frac{p(p-1)}{4}} \Gamma(\alpha) \Gamma\left(\alpha - \frac{1}{2}\right) \cdots \Gamma\left(\alpha - \frac{p-1}{2}\right)$$

for  $\text{Re}(\alpha) > \frac{p-1}{2}$ .

*Proof.* Note that since  $X_1$  is  $n \times p$ , the sum of squares of the  $np$  variables in  $X_1$  is given by

$$\text{Tr}(X_1^\top X_1) = \sum_{i=1}^n \sum_{j=1}^p x_{ij}^2.$$

Then

$$\begin{aligned} \int_{X_1} [\mathbf{d}X_1] e^{-\text{Tr}(X_1^\top X_1)} &= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e^{-\sum_{i=1}^n \sum_{j=1}^p x_{ij}^2} \prod_{i=1}^n \prod_{j=1}^p \mathbf{d}x_{ij} \\ &= \prod_{i=1}^n \prod_{j=1}^p \int_{-\infty}^{\infty} e^{-x_{ij}^2} \mathbf{d}x_{ij} \\ &= \pi^{\frac{np}{2}} \end{aligned}$$

by direct evaluation of the exponential integrals. Make the transformation as in Proposition 2.21:

$$X_1 = U_1 T_1 \implies X_1^\top X_1 = T_1^\top T_1,$$

where  $T_1$  is a real  $p \times p$  upper triangular matrix with distinct nonzero diagonal entries and  $U_1$  is a unique real  $n \times p$  semi-orthogonal matrix— $U_1^\top U_1 = \mathbb{1}_p$ , implying

$$\text{Tr}(X_1^\top X_1) = \text{Tr}(T_1^\top T_1) = \sum_{i \leq j} t_{ij}^2.$$

Note that  $\mathbf{d}X_1$  is available from Proposition 2.21. Now

$$\int_{X_1} [\mathbf{d}X_1] e^{-\text{Tr}(X_1^\top X_1)} = \left( \int_{T_1} \prod_{j=1}^p |t_{jj}|^{n-j} e^{-\sum_{i \leq j} t_{ij}^2} [\mathbf{d}T_1] \right) \left( \int_{\mathcal{O}(p, n)} \wedge_{j=1}^p \wedge_{i=j+1}^n \langle u_i, \mathbf{d}u_j \rangle \right).$$

But for  $0 < t_{jj} < \infty, -\infty < t_{ij} < \infty (i < j)$  and  $U_1$  unrestricted,

$$\int_{T_1} \prod_{j=1}^p |t_{jj}|^{n-j} e^{-\sum_{i \leq j} t_{ij}^2} [dT_1] = 2^{-p} \Gamma_p \left( \frac{n}{2} \right)$$

observing that for  $j = 1, \dots, p$ , the  $p$  integrals

$$\int_0^\infty |t_{jj}|^{n-j} e^{-t_{jj}^2} dt_{jj} = 2^{-1} \Gamma \left( \frac{n}{2} - \frac{j-1}{2} \right), n > j-1,$$

and each of the  $p(p-1)/2$  integrals

$$\int_{-\infty}^\infty e^{-t_{ij}^2} dt_{ij} = \sqrt{\pi}, i < j.$$

Thus the result that follows. □

**Theorem 2.23.** *Let  $X$  be a full-ranked and  $n \times n$  matrix of independent real entries and let  $X = UT$ , where  $T$  is a real  $n \times n$  upper triangular matrix with distinct nonzero diagonal entries and  $U$  is a unique real orthogonal matrix. Let  $u_j$  be the  $j$ -th column of  $U$  and  $du_j$  its differential. Then the volume content of the full orthogonal group  $\mathcal{O}(n)$  is given by*

$$\text{vol}(\mathcal{O}(n)) = \int_{\mathcal{O}(n)} \wedge [U^T dU] = \frac{2^n \pi^{\frac{n^2}{2}}}{\Gamma_n \left( \frac{n}{2} \right)} = \frac{2^n \pi^{\frac{n(n+1)}{4}}}{\prod_{k=1}^n \Gamma \left( \frac{k}{2} \right)}. \quad (2.33)$$

**Proposition 2.24.** *Let  $X$  be a  $p \times n (p \leq n)$  matrix of rank  $p$  and let  $X = TU_1^T$ , where  $T$  is a  $p \times p$  lower triangular matrix with distinct positive diagonal entries  $t_{jj} > 0, j = 1, \dots, p$  and  $U_1$  is a unique  $n \times p$  semi-orthogonal matrix,  $U_1^T U_1 = \mathbb{1}_p$ , all are of independent real entries. Let  $A = XX^T = TT^T$ . Then*

$$[dX] = 2^{-p} \det(A)^{\frac{n}{2} - \frac{p+1}{2}} [dA] \wedge_{j=1}^p \wedge_{i=j+1}^n \langle u_i, du_j \rangle.$$

*Proof.* Since  $A = TT^T$ , it follows that

$$[dA] = 2^p \left( \prod_{j=1}^p t_{jj}^{p+1-j} \right) [dT],$$

i.e.

$$[dT] = 2^{-p} \left( \prod_{j=1}^p t_{jj}^{-p+1+j} \right) [dA].$$

But

$$[dX] = \left( \prod_{j=1}^p |t_{jj}|^{n-j} \right) [dT] [dU_1],$$

where

$$[dU_1] = \prod_{j=1}^p \prod_{i=j+1}^n \langle u_i, du_j \rangle.$$

Note that  $\det(A) = \det(T)^2 = \prod_{j=1}^p t_{jj}^2$ . The desired conclusion is obtained.  $\square$

**Proposition 2.25.** *Let  $X$  be a  $m \times n$  matrix of rank  $m$  ( $m \leq n$ ),  $T = [t_{jk}]$  a  $m \times m$  lower triangular matrix with  $t_{jj} > 0, j = 1, \dots, m$  and  $L$  a  $n \times m$  matrix satisfying  $L^\top L = \mathbb{1}_m$ , where the matrices are of independent real entries. Then show that, if  $X = TL^\top$ , then*

$$[dX] = \left( \prod_{j=1}^m t_{jj}^{n-j} \right) [dT][d\hat{L}] \quad (2.34)$$

where

$$d\hat{L} = \prod_{j=1}^m \prod_{i=j+1}^n \langle l_i, dl_j \rangle,$$

$l_j$  is the  $j$ -th column of  $\hat{L} = [L \ L_1] \in \mathcal{O}(n)$ ;  $dl_i$  the differential of the  $i$ -th column of  $L$ .

**Proposition 2.26** (Polar decomposition). *Let  $X$  be a  $m \times n$  ( $m \leq n$ ) matrix,  $S$  a  $m \times m$  symmetric positive definite matrix and  $L$  a  $n \times m$  matrix with  $L^\top L = \mathbb{1}_m$ , all are of independent real entries. Then show that, if  $X = \sqrt{S}L^\top$ , then*

$$[dX] = \left( \frac{1}{2} \right)^m (\det(S))^{\frac{n-m-1}{2}} [dS][d\hat{L}] \quad (2.35)$$

where  $d\hat{L}$  is defined in Proposition 2.25.

*Proof.* Now if  $X = \sqrt{S}L^\top$  and  $L^\top L = \mathbb{1}_m$ , then  $XX^\top = S$ . By Proposition 2.25, we have  $X = TL^\top$ , where  $T = [t_{jk}]$  is a  $m \times m$  lower triangular matrix with  $t_{jj} > 0, j = 1, \dots, m$  and  $L$  a  $n \times m$  matrix satisfying  $L^\top L = \mathbb{1}_m$ . Denote  $\hat{L} = [L \ L_1] \in \mathcal{O}(n)$ . Hence

$$[dX] = \left( \prod_{j=1}^m t_{jj}^{n-j} \right) [dT][d\hat{L}].$$

It also holds that  $S = TT^\top$  implies

$$[dS] = 2^m \left( \prod_{j=1}^m t_{jj}^{m+1-j} \right) [dT].$$

Both expressions indicate that

$$[dX] = 2^{-m} \left( \prod_{j=1}^m t_{jj}^{n-m-1} \right) [dS][d\hat{L}].$$

Since  $\det(S) = \det(TT^\top) = \prod_{j=1}^m t_{jj}^2$ , it follows that

$$[dX] = \left( \frac{1}{2} \right)^m (\det(S))^{\frac{n-m-1}{2}} [dS][d\hat{L}].$$

We are done.  $\square$

**Proposition 2.27.** *With the same notations as in Proposition 2.26, it holds that*

$$\int_{L^\top L = \mathbb{1}_m} [\mathrm{d}\hat{L}] = \frac{2^m \pi^{\frac{mn}{2}}}{\Gamma_m\left(\frac{n}{2}\right)} \quad (2.36)$$

and for  $m = n$

$$\int_{\mathcal{O}(n)} [\mathrm{d}\hat{V}] = \frac{2^n \pi^{\frac{n^2}{2}}}{\Gamma_n\left(\frac{n}{2}\right)}. \quad (2.37)$$

Define the normalized orthogonal measures as

$$\mathrm{d}\mu(V) := \left( \frac{\Gamma_n\left(\frac{n}{2}\right)}{2^n \pi^{\frac{n^2}{2}}} \right) [\mathrm{d}\hat{V}] = \left( \frac{\Gamma_n\left(\frac{n}{2}\right)}{2^n \pi^{\frac{n^2}{2}}} \right) \prod_{i>j} \langle v_i, \mathrm{d}v_j \rangle \quad (2.38)$$

or

$$\mathrm{d}\mu(V) := \left( \frac{\Gamma_n\left(\frac{n}{2}\right)}{2^n \pi^{\frac{n^2}{2}}} \right) [\mathrm{d}G], \quad (2.39)$$

where  $\mathrm{d}G = V^\top \mathrm{d}V$  for  $V = [v_1, \dots, v_n] \in \mathcal{O}(n)$ . It holds that  $[\mathrm{d}\hat{V}] = \wedge(V^\top \mathrm{d}V)$  is invariant under simultaneous translations  $V \rightarrow UVW$ , where  $U, W \in \mathcal{O}(n)$ . That is,  $\mathrm{d}\mu(V)$  is an invariant measure under both left and right translations, i.e. Haar measure over  $\mathcal{O}(n)$ .

*Proof.* We know that

$$\int_X [\mathrm{d}X] e^{-\mathrm{Tr}(XX^\top)} = \pi^{\frac{mn}{2}}.$$

From Proposition 2.26, via the transformation  $X = \sqrt{S}L^\top$ , we see that

$$\int_X [\mathrm{d}X] e^{-\mathrm{Tr}(XX^\top)} = \left( \frac{1}{2} \right)^m \int_{S>0} \det(S)^{\frac{n-m-1}{2}} e^{-\mathrm{Tr}(S)} [\mathrm{d}S] \times \int_{L^\top L = \mathbb{1}_m} [\mathrm{d}\hat{L}].$$

We also see from the definition of  $\Gamma_p(\alpha)$  that

$$\Gamma_m\left(\frac{n}{2}\right) = \int_{S>0} \det(S)^{\frac{n}{2} - \frac{m+1}{2}} e^{-\mathrm{Tr}(S)} [\mathrm{d}S].$$

Then

$$\int_{L^\top L = \mathbb{1}_m} [\mathrm{d}\hat{L}] = \frac{2^m \pi^{\frac{mn}{2}}}{\Gamma_m\left(\frac{n}{2}\right)}.$$

For  $m = n$ , the result follows easily.

For fixed  $U, W \in \mathcal{O}(n)$ , we have

$$(UVW)^T d(UVW) = (W^T V^T U^T) (U \cdot dV \cdot W) = W^T \cdot dG \cdot W,$$

implying that

$$\wedge [(UVW)^T d(UVW)] = [dG].$$

That is,  $d\mu(V) = d\mu(UVW)$  for all  $U, V, W \in \mathcal{O}(n)$ ,  $d\mu(V)$  is an invariant measure under both left and right translations over  $\mathcal{U}(n)$ .  $\square$

### 3 Volumes of unitary groups

#### 3.1 Preliminary

In Section 2, we dealt with matrices where the entries are either *real* constants or *real* variables. Here we consider the matrices whose entries are complex quantities. When the matrices are real, we will use the same notations as in Section 2. In the complex case, the matrix variable  $X$  will be denoted by  $\tilde{X}$  to indicate that the entries in  $X$  are complex variables so that the entries of theorems in Section 3 will not be confused with those in Section 2. The complex conjugate of a matrix  $\tilde{A}$  will be denoted by  $\overline{\tilde{A}}$  and the conjugate transpose by  $\tilde{A}^*$ . The determinant of  $\tilde{A}$  will be denoted by  $\det(\tilde{A})$ . The absolute value of a scalar  $a$  will also be denoted by  $|a|$ . The *wedge product of differentials* in  $\tilde{X}$  will be denoted by  $[d\tilde{X}]$  and the *matrix of differentials* by  $d\tilde{X}$ .

It is assumed that the reader is familiar with the basic properties of real and complex matrices. Some properties of complex matrices will be listed here for convenience.

A matrix  $\tilde{X}$  with complex elements can always be written as  $\tilde{X} = X_1 + \sqrt{-1}X_2$  where  $X_1 = \text{Re}(\tilde{X})$  and  $X_2 = \text{Im}(\tilde{X})$  are real matrices. Let us examine the wedge product of the differentials in  $\tilde{X}$ . In general, there are  $n^2$  real variables in  $X_1$  and another  $n^2$  real variables and the wedge product of the differentials will be denoted by the following:

**Notation 1:**

$$[d\tilde{X}] := [dX_1][dX_2] \text{ or } [d\tilde{X}] := \left[ d \left( \text{Re}(\tilde{X}) \right) \right] \left[ d \left( \text{Im}(\tilde{X}) \right) \right] \quad (3.1)$$

where  $[dX_1]$  is the wedge product in  $dX_1$  and  $[dX_2]$  is the wedge product in  $dX_2$ . In this notation an empty product is interpreted as unity. That is, when the matrix  $\tilde{X}$  is real then  $X_2$  is null and  $[d\tilde{X}] := [dX_1]$ . If  $\tilde{X}$  is a hermitian matrix, then  $X_1$  is *symmetric* and  $X_2$  is *skew symmetric*, and in this case

$$[dX_1] = \wedge_{j \geq k} dx_{jk}^{(1)} \text{ and } [dX_2] = \wedge_{j > k} dx_{jk}^{(2)} \quad (3.2)$$

where  $X_1 = [x_{jk}^{(1)}]$  and  $X_2 = [x_{jk}^{(2)}]$ . If  $\tilde{Y}$  is a scalar function of  $\tilde{X} = X_1 + \sqrt{-1}X_2$  then  $\tilde{Y}$  can be written as  $\tilde{Y} = Y_1 + \sqrt{-1}Y_2$  where  $Y_1$  and  $Y_2$  are real. Thus if  $\tilde{Y} = F(\tilde{X})$  it is a transformation of  $(X_1, X_2)$  to  $(Y_1, Y_2)$  or where  $(Y_1, Y_2)$  is written as a function of  $(X_1, X_2)$  then we will use the following notation for the Jacobian in the complex case.

**Notation 2: (Jacobians in the complex case).**  $J(Y_1, Y_2 : X_1, X_2)$ : Jacobian of the transformation where  $Y_1$  and  $Y_2$  are written as functions of  $X_1$  and  $X_2$  or where  $\tilde{Y} = Y_1 + \sqrt{-1}Y_2$  is a function of  $\tilde{X} = X_1 + \sqrt{-1}X_2$ .

**Lemma 3.1.** Consider a matrix  $\tilde{A} \in \mathbb{C}^{n \times n}$  and  $(2n) \times (2n)$  matrices  $B$  and  $C$  where

$$\tilde{A} = A_1 + \sqrt{-1}A_2, B = \begin{bmatrix} A_1 & A_2 \\ -A_2 & A_1 \end{bmatrix}, C = \begin{bmatrix} A_1 & -A_2 \\ A_2 & A_1 \end{bmatrix}$$

where  $A_1, A_2 \in \mathbb{R}^{n \times n}$ . Then for  $\det(A_1) \neq 0$

$$|\det(\tilde{A})| = |\det(B)|^{\frac{1}{2}} = |\det(C)|^{\frac{1}{2}}. \quad (3.3)$$

*Proof.* Let  $\det(A) = a + \sqrt{-1}b$  where  $a$  and  $b$  are real scalars. Then the absolute value is available as  $\sqrt{(a + \sqrt{-1}b)(a - \sqrt{-1}b)}$ . If  $\det(A_1 + \sqrt{-1}A_2) = a + \sqrt{-1}b$ , then  $\det(A_1 - \sqrt{-1}A_2) = a - \sqrt{-1}b$ . Hence

$$\begin{aligned} (a + \sqrt{-1}b)(a - \sqrt{-1}b) &= \det(A_1 + \sqrt{-1}A_2) \det(A_1 - \sqrt{-1}A_2) \\ &= \det \left( \begin{bmatrix} A_1 + \sqrt{-1}A_2 & 0 \\ 0 & A_1 - \sqrt{-1}A_2 \end{bmatrix} \right). \end{aligned}$$

Adding the last  $n$  columns to the first  $n$  columns and then adding the last  $n$  rows to the first  $n$  rows we have

$$\det \left( \begin{bmatrix} A_1 + \sqrt{-1}A_2 & 0 \\ 0 & A_1 - \sqrt{-1}A_2 \end{bmatrix} \right) = \det \left( \begin{bmatrix} 2A_1 & A_1 - \sqrt{-1}A_2 \\ A_1 - \sqrt{-1}A_2 & A_1 - \sqrt{-1}A_2 \end{bmatrix} \right).$$

Using similar steps we have

$$\begin{aligned} \det \left( \begin{bmatrix} 2A_1 & A_1 - \sqrt{-1}A_2 \\ A_1 - \sqrt{-1}A_2 & A_1 - \sqrt{-1}A_2 \end{bmatrix} \right) &= \det \left( \begin{bmatrix} 2A_1 & A_1 - \sqrt{-1}A_2 \\ -\sqrt{-1}A_2 & \frac{1}{2}A_1 - \frac{1}{2}\sqrt{-1}A_2 \end{bmatrix} \right) \\ &= \det \left( \begin{bmatrix} 2A_1 & -\sqrt{-1}A_2 \\ -\sqrt{-1}A_2 & \frac{1}{2}A_1 \end{bmatrix} \right) \\ &= \det(A_1) \det(A_1 + A_2 A_1^{-1} A_2) \\ &= \det \left( \begin{bmatrix} A_1 & A_2 \\ -A_2 & A_1 \end{bmatrix} \right) = \det \left( \begin{bmatrix} A_1 & -A_2 \\ A_2 & A_1 \end{bmatrix} \right). \end{aligned}$$

by evaluating as the determinant of partitioned matrices. Thus the absolute value of  $\det(\tilde{A})$  is given by

$$\left| \det(\tilde{A}) \right| = \sqrt{\det(A_1) \det(A_1 + A_2 A_1^{-1} A_2)} = |\det(B)|^{\frac{1}{2}} = |\det(C)|^{\frac{1}{2}}.$$

This establishes the result.  $\square$

**Remark 3.2.** Now we denote  $A_1 = \operatorname{Re}(\tilde{A})$  and  $A_2 = \operatorname{Im}(\tilde{A})$ . Clearly both  $\operatorname{Re}(\tilde{A})$  and  $\operatorname{Im}(\tilde{A})$  are real matrices. Each complex matrix  $\tilde{A} := \operatorname{Re}(\tilde{A}) + \sqrt{-1}\operatorname{Im}(\tilde{A})$  can be represented faithfully as a block-matrix

$$\tilde{A} \longrightarrow \begin{bmatrix} \operatorname{Re}(\tilde{A}) & -\operatorname{Im}(\tilde{A}) \\ \operatorname{Im}(\tilde{A}) & \operatorname{Re}(\tilde{A}) \end{bmatrix}. \quad (3.4)$$

Thus

$$\tilde{A}^* \longrightarrow \begin{bmatrix} \operatorname{Re}(\tilde{A})^\top & \operatorname{Im}(\tilde{A})^\top \\ -\operatorname{Im}(\tilde{A})^\top & \operatorname{Re}(\tilde{A})^\top \end{bmatrix}. \quad (3.5)$$

Then  $\tilde{Y} = \tilde{A}\tilde{X}$  can be rewritten as, via block-matrix technique,

$$\begin{bmatrix} \operatorname{Re}(\tilde{Y}) & -\operatorname{Im}(\tilde{Y}) \\ \operatorname{Im}(\tilde{Y}) & \operatorname{Re}(\tilde{Y}) \end{bmatrix} = \begin{bmatrix} \operatorname{Re}(\tilde{A}) & -\operatorname{Im}(\tilde{A}) \\ \operatorname{Im}(\tilde{A}) & \operatorname{Re}(\tilde{A}) \end{bmatrix} \begin{bmatrix} \operatorname{Re}(\tilde{X}) & -\operatorname{Im}(\tilde{X}) \\ \operatorname{Im}(\tilde{X}) & \operatorname{Re}(\tilde{X}) \end{bmatrix}. \quad (3.6)$$

From the above, we see that the mentioned representation is an injective ring homomorphism which is continuous. sometimes we use the following representation:

$$\begin{bmatrix} \operatorname{Re}(\tilde{Y}) \\ \operatorname{Im}(\tilde{Y}) \end{bmatrix} = \begin{bmatrix} \operatorname{Re}(\tilde{A}) & -\operatorname{Im}(\tilde{A}) \\ \operatorname{Im}(\tilde{A}) & \operatorname{Re}(\tilde{A}) \end{bmatrix} \begin{bmatrix} \operatorname{Re}(\tilde{X}) \\ \operatorname{Im}(\tilde{X}) \end{bmatrix}. \quad (3.7)$$

Lemma 3.1 can be reexpressed as

$$\left| \det \left( \begin{bmatrix} \operatorname{Re}(\tilde{A}) & -\operatorname{Im}(\tilde{A}) \\ \operatorname{Im}(\tilde{A}) & \operatorname{Re}(\tilde{A}) \end{bmatrix} \right) \right| = |\det(\tilde{A})|^2 = |\det(\tilde{A}\tilde{A}^*)|. \quad (3.8)$$

**Proposition 3.3.** Let  $\tilde{X}, \tilde{Y} \in \mathbb{C}^n$  be of  $n$  independent complex variables each,  $\tilde{A} \in \mathbb{C}^{n \times n}$  a nonsingular matrix of constants. If  $\tilde{Y} = \tilde{A}\tilde{X}$ , then

$$[\mathrm{d}\tilde{Y}] = |\det(\tilde{A}\tilde{A}^*)| [\mathrm{d}\tilde{X}]. \quad (3.9)$$

If  $\tilde{Y}^* = \tilde{X}^* \tilde{A}^*$ , then

$$[\mathrm{d}\tilde{Y}^*] = (-1)^n |\det(\tilde{A}\tilde{A}^*)| [\mathrm{d}\tilde{X}]. \quad (3.10)$$

*Proof.* Let  $\tilde{X} = X_1 + \sqrt{-1}X_2$ , where  $X_m \in \mathbb{R}^n, m = 1, 2$ . Let  $\tilde{Y} = Y_1 + \sqrt{-1}Y_2$ , where  $Y_m \in \mathbb{R}^n, m = 1, 2$  are real.  $\tilde{Y} = \tilde{A}\tilde{X}$  implies that  $Y_m = AX_m, m = 1, 2$  if  $\tilde{A} = A$  is real. This transformation is such that the  $2n$  real variables in  $(Y_1, Y_2)$  are written as functions of the  $2n$  real variables in  $(X_1, X_2)$ . Let

$$\begin{aligned} X_1^\top &= [x_{11}, \dots, x_{n1}], & X_2^\top &= [x_{12}, \dots, x_{n2}], \\ Y_1^\top &= [y_{11}, \dots, y_{n1}], & Y_2^\top &= [y_{12}, \dots, y_{n2}]. \end{aligned}$$

Then the Jacobian is the determinant of the following matrix of partial derivatives:

$$\frac{\partial(Y_1, Y_2)}{\partial(X_1, X_2)} = \frac{\partial(y_{11}, \dots, y_{n1}, y_{12}, \dots, y_{n2})}{\partial(x_{11}, \dots, x_{n1}, x_{12}, \dots, x_{n2})}.$$

Note that

$$\frac{\partial(Y_1, Y_2)}{\partial(X_1, X_2)} = \frac{\partial(y_{11}, \dots, y_{n1}, y_{12}, \dots, y_{n2})}{\partial(x_{11}, \dots, x_{n1}, x_{12}, \dots, x_{n2})}.$$

Note that

$$\begin{aligned} \frac{\partial Y_1}{\partial X_1} &= \frac{\partial(y_{11}, \dots, y_{n1})}{\partial(x_{11}, \dots, x_{n1})} = A, \\ \frac{\partial Y_1}{\partial X_2} &= 0 = \frac{\partial Y_2}{\partial X_1}, \\ \frac{\partial Y_2}{\partial X_2} &= A. \end{aligned}$$

Thus the Jacobian is

$$J = \det \left( \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix} \right) = \det(A)^2.$$

If  $\tilde{A}$  is complex, then let  $\tilde{A} = A_1 + \sqrt{-1}A_2$  where  $A_1$  and  $A_2$  are real. Then

$$\begin{aligned} \tilde{Y} &= Y_1 + \sqrt{-1}Y_2 = (A_1 + \sqrt{-1}A_2)(X_1 + \sqrt{-1}X_2) \\ &= (A_1X_1 - A_2X_2) + \sqrt{-1}(A_1X_2 + A_2X_1) \end{aligned}$$

implies that

$$Y_1 = A_1X_1 - A_2X_2, \quad Y_2 = A_1X_2 + A_2X_1.$$

Then

$$\begin{aligned} \frac{\partial Y_1}{\partial X_1} &= A_1, & \frac{\partial Y_1}{\partial X_2} &= -A_2, \\ \frac{\partial Y_2}{\partial X_1} &= A_2, & \frac{\partial Y_2}{\partial X_2} &= A_1. \end{aligned}$$

Thus the Jacobian is

$$J = \det \left( \begin{bmatrix} A_1 & -A_2 \\ A_2 & A_1 \end{bmatrix} \right) = |\det(\tilde{A})|^2 = |\det(\tilde{A}\tilde{A}^*)|,$$

which establishes the result. The second result follows by noting that  $[\mathrm{d}\tilde{Y}^*] = [\mathrm{d}\tilde{Y}_1](-1)^n[\mathrm{d}\tilde{Y}_2] = (-1)^n[\mathrm{d}\tilde{Y}]$ .  $\square$

**Proposition 3.4.** *Let  $\tilde{X}, \tilde{Y} \in \mathbb{C}^{m \times n}$  of  $mn$  independent complex variables each. Let  $\tilde{A} \in \mathbb{C}^{m \times m}$  and  $\tilde{B} \in \mathbb{C}^{n \times n}$  nonsingular matrices of constants. If  $\tilde{Y} = \tilde{A}\tilde{X}\tilde{B}$ , then*

$$[\mathrm{d}\tilde{Y}] = |\det(\tilde{A}\tilde{A}^*)|^n |\det(\tilde{B}\tilde{B}^*)|^m [\mathrm{d}\tilde{X}]. \quad (3.11)$$

*Proof.* Let  $\tilde{Y} = Y_1 + \sqrt{-1}Y_2$  and  $\tilde{X} = X_1 + \sqrt{-1}X_2$ . Indeed, let  $\tilde{Y} = [\tilde{Y}_1, \dots, \tilde{Y}_n]$  and  $\tilde{X} = [\tilde{X}_1, \dots, \tilde{X}_n]$ , then  $\tilde{Y}_j = \tilde{A}\tilde{X}_j, j = 1, \dots, n$  when  $\tilde{Y} = \tilde{A}\tilde{X}$ . Thus  $[\mathrm{d}\tilde{Y}_j] = |\det(\tilde{A}\tilde{A}^*)| [\mathrm{d}\tilde{X}_j]$  for each  $j$ , therefore, ignoring the signs,

$$[\mathrm{d}\tilde{Y}] = \prod_{j=1}^n [\mathrm{d}\tilde{Y}_j] = |\det(\tilde{A}\tilde{A}^*)|^n [\mathrm{d}\tilde{X}].$$

Denoting  $\tilde{A} = A_1 + \sqrt{-1}A_2$ , the determinant is

$$J = \det \left( \begin{bmatrix} A_1 & -A_2 \\ A_2 & A_1 \end{bmatrix} \right)^n.$$

Hence the Jacobian in this case, denoting  $\tilde{B} = B_1 + \sqrt{-1}B_2$ , is given by

$$J = \det \left( \begin{bmatrix} B_1 & B_2 \\ -B_2 & B_1 \end{bmatrix} \right)^m.$$

For establishing our result, write  $\tilde{Y} = \tilde{A}\tilde{Z}$  where  $\tilde{Z} = \tilde{X}\tilde{B}$ . That is,

$$[\mathrm{d}\tilde{Y}] = |\det(\tilde{A}\tilde{A}^*)|^n [\mathrm{d}\tilde{Z}] = |\det(\tilde{A}\tilde{A}^*)|^n |\det(\tilde{B}\tilde{B}^*)|^m [\mathrm{d}\tilde{X}].$$

This completes the proof.  $\square$

**Remark 3.5.** Another approach to the fact that  $[\mathrm{d}\tilde{Y}] = |\det(\tilde{A}\tilde{A}^*)|^n [\mathrm{d}\tilde{Z}]$ , where  $\tilde{Y} = \tilde{A}\tilde{Z}$ , is described as follows:

$$\begin{cases} \operatorname{Re}(\tilde{Y}) &= \operatorname{Re}(\tilde{A})\operatorname{Re}(\tilde{Z}) - \operatorname{Im}(\tilde{A})\operatorname{Im}(\tilde{Z}) \\ \operatorname{Im}(\tilde{Y}) &= \operatorname{Im}(\tilde{A})\operatorname{Re}(\tilde{Z}) + \operatorname{Re}(\tilde{A})\operatorname{Im}(\tilde{Z}) \end{cases}$$

leading to

$$\frac{\partial (\operatorname{Re}(\tilde{Y}), \operatorname{Im}(\tilde{Y}))}{\partial (\operatorname{Re}(\tilde{Z}), \operatorname{Im}(\tilde{Z}))} = \begin{bmatrix} \operatorname{Re}(\tilde{A})^{(n)} & -\operatorname{Im}(\tilde{A})^{(n)} \\ \operatorname{Im}(\tilde{A})^{(n)} & \operatorname{Re}(\tilde{A})^{(n)} \end{bmatrix},$$

where

$$\operatorname{Re}(\tilde{A})^{(n)} := \begin{bmatrix} \operatorname{Re}(\tilde{A}) & & \\ & \ddots & \\ & & \operatorname{Re}(\tilde{A}) \end{bmatrix}, \quad \operatorname{Im}(\tilde{A})^{(n)} := \begin{bmatrix} \operatorname{Im}(\tilde{A}) & & \\ & \ddots & \\ & & \operatorname{Im}(\tilde{A}) \end{bmatrix}.$$

Then the Jacobian of this transformation can be computed as

$$J \left( \operatorname{Re}(\tilde{Y}), \operatorname{Im}(\tilde{Y}) : \operatorname{Re}(\tilde{Z}), \operatorname{Im}(\tilde{Z}) \right) = \det \left( \begin{bmatrix} \operatorname{Re}(\tilde{A})^{(n)} & -\operatorname{Im}(\tilde{A})^{(n)} \\ \operatorname{Im}(\tilde{A})^{(n)} & \operatorname{Re}(\tilde{A})^{(n)} \end{bmatrix} \right) \quad (3.12)$$

$$= \left| \det \left( \operatorname{Re}(\tilde{A})^{(n)} + \sqrt{-1} \operatorname{Im}(\tilde{A})^{(n)} \right) \right|^2. \quad (3.13)$$

That is,

$$J \left( \operatorname{Re}(\tilde{Y}), \operatorname{Im}(\tilde{Y}) : \operatorname{Re}(\tilde{Z}), \operatorname{Im}(\tilde{Z}) \right) = \left| \det \left( \tilde{A}^{(n)} \right) \right|^2 = \left| \det(\tilde{A}) \right|^{2n} = \left| \det(\tilde{A} \tilde{A}^*) \right|^n.$$

**Proposition 3.6.** *Let  $\tilde{X}, \tilde{A}, \tilde{B} \in \mathbb{C}^{n \times n}$  be lower triangular matrices where  $\tilde{X}$  is matrix of  $\frac{n(n+1)}{2}$  independent complex variables,  $\tilde{A}, \tilde{B}$  are nonsingular matrices of constants. Then*

$$\tilde{Y} = \tilde{X} + \tilde{X}^\top \implies [\mathrm{d}\tilde{Y}] = 2^{2n} [\mathrm{d}\tilde{X}], \quad (3.14)$$

$$\implies [\mathrm{d}\tilde{Y}] = 2^n [\mathrm{d}\tilde{X}] \text{ if the } \tilde{x}_{jj}'\text{'s are real}; \quad (3.15)$$

$$\tilde{Y} = \tilde{A} \tilde{X} \implies [\mathrm{d}\tilde{Y}] = \left( \prod_{j=1}^n |\tilde{a}_{jj}|^{2j} \right) [\mathrm{d}\tilde{X}], \quad (3.16)$$

$$\implies [\mathrm{d}\tilde{Y}] = \left( \prod_{j=1}^n |\tilde{a}_{jj}|^{2j-1} \right) [\mathrm{d}\tilde{X}] \text{ if the } \tilde{a}_{jj}'\text{'s and } \tilde{x}_{jj}'\text{'s are real}; \quad (3.17)$$

$$\tilde{Y} = \tilde{X} \tilde{B} \implies [\mathrm{d}\tilde{Y}] = \left( \prod_{j=1}^n |\tilde{b}_{jj}|^{2(n-j+1)} \right) [\mathrm{d}\tilde{X}], \quad (3.18)$$

$$\implies [\mathrm{d}\tilde{Y}] = \left( \prod_{j=1}^n |\tilde{b}_{jj}|^{2(n-j)+1} \right) [\mathrm{d}\tilde{X}] \text{ if the } \tilde{b}_{jj}'\text{'s and } \tilde{x}_{jj}'\text{'s are real}; \quad (3.19)$$

*Proof.* Results (3.14) and (3.15) are trivial. Indeed, note that

$$\tilde{y}_{jk} = \begin{cases} 2\tilde{x}_{jj}, & \text{if } j = k \\ \tilde{x}_{jk}, & \text{if } j > k \end{cases}.$$

By the definition, ignoring the sign, we have

$$[\mathrm{d}\tilde{Y}] = \wedge_{j \geq k} \mathrm{d}\tilde{y}_{jk} = \wedge_{j=1}^n \mathrm{d}\tilde{y}_{jj} \wedge_{j > k} \mathrm{d}\tilde{y}_{jk},$$

where  $d\tilde{y}_{jk} := dy_{jk}^{(1)} dy_{jk}^{(2)}$  for  $\tilde{y}_{jk} = y_{jk}^{(1)} + \sqrt{-1}y_{jk}^{(2)}$ . So for  $j = 1, \dots, n$ , we get  $y_{jj}^{(m)} = 2x_{jj}^{(m)}$ ,  $m = 1, 2$ . Hence the result. If  $\tilde{x}_{jj}$ 's are real, the result follows easily by definition. Let

$$\begin{aligned}\tilde{Y} &= Y_1 + \sqrt{-1}Y_2, \tilde{X} = X_1 + \sqrt{-1}X_2, \tilde{A} = A_1 + \sqrt{-1}A_2, \tilde{B} = B_1 + \sqrt{-1}B_2, \\ Y_m &= [y_{jk}^{(m)}], X_m = [x_{jk}^{(m)}], A_m = [a_{jk}^{(m)}], B_m = [b_{jk}^{(m)}], m = 1, 2.\end{aligned}$$

where  $Y_m, X_m, A_m, B_m, m = 1, 2$  are all real.

When  $\tilde{Y} = \tilde{A}\tilde{X}$  we have  $Y_1 = A_1X_1 - A_2X_2$  and  $Y_2 = A_1X_2 + A_2X_1$ . The matrix of partial derivative of  $Y_1$  with respect to  $X_1$ , that is  $\frac{\partial Y_1}{\partial X_1}$ , can be seen to be a lower triangular matrix with  $a_{jj}^{(1)}$  repeated  $j$  times,  $j = 1, \dots, n$ , on the diagonal. Let this matrix be denoted by  $G_1$ . That is,

$$\frac{\partial Y_1}{\partial X_1} = \frac{\partial Y_2}{\partial X_2} := G_1 = \begin{bmatrix} A_1 & & & \\ & A_1[\hat{1}|\hat{1}] & & \\ & & \ddots & \\ & & & A_1[\hat{1} \cdots \widehat{n-1} | \hat{1} \cdots \widehat{n-1}] \end{bmatrix}.$$

Let  $G_2$  be a matrix of the same structure with  $a_{jj}^{(2)}$ 's on the diagonal. Similarly,

$$-\frac{\partial Y_1}{\partial X_2} = \frac{\partial Y_2}{\partial X_1} := G_2 = \begin{bmatrix} A_2 & & & \\ & A_2[\hat{1}|\hat{1}] & & \\ & & \ddots & \\ & & & A_2[\hat{1} \cdots \widehat{n-1} | \hat{1} \cdots \widehat{n-1}] \end{bmatrix}.$$

Then the Jacobian matrix is given by

$$\frac{\partial(Y_1, Y_2)}{\partial(X_1, X_2)} = \begin{bmatrix} G_1 & -G_2 \\ G_2 & G_1 \end{bmatrix}.$$

Let  $\tilde{G} = G_1 + \sqrt{-1}G_2$ . Then

$$\tilde{G} = \begin{bmatrix} \tilde{A} & & & \\ & \tilde{A}[\hat{1}|\hat{1}] & & \\ & & \ddots & \\ & & & \tilde{A}[\hat{1} \cdots \widehat{n-1} | \hat{1} \cdots \widehat{n-1}] \end{bmatrix},$$

where

$$\begin{aligned}\tilde{A} &= A_1 + \sqrt{-1}A_2, \tilde{A}[\hat{1}|\hat{1}] = A_1[\hat{1}|\hat{1}] + \sqrt{-1}A_2[\hat{1}|\hat{1}], \dots, \\ \tilde{A}[\hat{1} \cdots \widehat{n-1} | \hat{1} \cdots \widehat{n-1}] &= A_1[\hat{1} \cdots \widehat{n-1} | \hat{1} \cdots \widehat{n-1}] + \sqrt{-1}A_2[\hat{1} \cdots \widehat{n-1} | \hat{1} \cdots \widehat{n-1}].\end{aligned}$$

Thus

$$\det \left( \frac{\partial(Y_1, Y_2)}{\partial(X_1, X_2)} \right) = \det \left( \begin{bmatrix} G_1 & -G_2 \\ G_2 & G_1 \end{bmatrix} \right).$$

From Lemma 3.1, the determinant is available as  $\left| \det(\tilde{G}) \right|^2$  where  $\tilde{G} = G_1 + \sqrt{-1}G_2$ . Since  $\tilde{G}$  is triangular the absolute value of the determinant is given by

$$\begin{aligned} \left| \det(\tilde{G}) \right|^2 &= \left| \det(\tilde{A}) \right|^2 \left| \det(\tilde{A}[\hat{1}|\hat{1}]) \right|^2 \cdots \left| \det(\tilde{A}[\hat{1} \cdots \widehat{n-1}|\hat{1} \cdots \widehat{n-1}]) \right|^2 \\ &= \prod_{j=1}^n (|a_{jj}|^2)^j. \end{aligned}$$

This establishes (3.16). Another approach is presented also: Let  $\tilde{Y} = [\tilde{Y}_1, \dots, \tilde{Y}_n]$ , where  $\tilde{Y}_j, j = 1, \dots, n$ , is the  $j$ -th column of the matrix  $\tilde{Y}$ . Similarly for  $\tilde{X} = [\tilde{X}_1, \dots, \tilde{X}_n]$ . Now  $\tilde{Y} = \tilde{A}\tilde{X}$  implies that

$$\tilde{Y}_j = \tilde{A}\tilde{X}_j, \quad j = 1, \dots, n.$$

That is,

$$\begin{bmatrix} \tilde{y}_{11} \\ \tilde{y}_{21} \\ \vdots \\ \tilde{y}_{n1} \end{bmatrix} = \tilde{A} \begin{bmatrix} \tilde{x}_{11} \\ \tilde{x}_{21} \\ \vdots \\ \tilde{x}_{n1} \end{bmatrix}, \quad \begin{bmatrix} 0 \\ \tilde{y}_{22} \\ \vdots \\ \tilde{y}_{n2} \end{bmatrix} = \tilde{A} \begin{bmatrix} 0 \\ \tilde{x}_{22} \\ \vdots \\ \tilde{x}_{n2} \end{bmatrix}, \dots, \quad \begin{bmatrix} 0 \\ 0 \\ \vdots \\ \tilde{y}_{nn} \end{bmatrix} = \tilde{A} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ \tilde{x}_{nn} \end{bmatrix}.$$

Since  $\tilde{Y}, \tilde{X}, \tilde{A}$  are all lower triangular, it follows that

$$\begin{aligned} \begin{bmatrix} \tilde{y}_{11} \\ \tilde{y}_{21} \\ \vdots \\ \tilde{y}_{n1} \end{bmatrix} &= \tilde{A} \begin{bmatrix} \tilde{x}_{11} \\ \tilde{x}_{21} \\ \vdots \\ \tilde{x}_{n1} \end{bmatrix}, \quad \begin{bmatrix} \tilde{y}_{22} \\ \vdots \\ \tilde{y}_{n2} \end{bmatrix} = \tilde{A}[\hat{1}|\hat{1}] \begin{bmatrix} \tilde{x}_{22} \\ \vdots \\ \tilde{x}_{n2} \end{bmatrix}, \dots, \\ \tilde{y}_{nn} &= \tilde{A}[\hat{1} \cdots \widehat{n-1}|\hat{1} \cdots \widehat{n-1}] \tilde{x}_{nn}, \end{aligned}$$

where  $A[\hat{i}\hat{j}|\hat{i}\hat{j}]$  stands for a matrix obtained from deleting the  $i, j$ -th rows and columns of  $\tilde{A}$ , respectively. We can now draw the conclusion that

$$[\mathrm{d}\tilde{Y}_j] = \left| \det(A[\hat{1} \cdots \widehat{j-1}|\hat{1} \cdots \widehat{j-1}]) A[\hat{1} \cdots \widehat{j-1}|\hat{1} \cdots \widehat{j-1}]^* \right| [\mathrm{d}\tilde{X}_j],$$

that indicates that

$$\begin{aligned}
[d\tilde{Y}] &= \prod_{j=1}^n [d\tilde{Y}_j] = \prod_{j=1}^n \left| \det(A[\hat{1} \dots \widehat{j-1}] | \hat{1} \dots \widehat{j-1}) A[\hat{1} \dots \widehat{j-1}] | \hat{1} \dots \widehat{j-1}]^* \right| [d\tilde{X}_j] \\
&= \left| \det(\tilde{A}\tilde{A}^*) \right| \left| \det(\tilde{A}[\hat{1} | \hat{1}] \tilde{A}[\hat{1} | \hat{1}]^*) \right| \cdots |\tilde{a}_{nn} \tilde{a}_{nn}^*| [d\tilde{X}] \\
&= |\tilde{a}_{11} \tilde{a}_{22} \cdots \tilde{a}_{nn}|^2 \times |\tilde{a}_{22} \tilde{a}_{33} \cdots \tilde{a}_{nn}|^2 \times \cdots \times |\tilde{a}_{nn}|^2 [d\tilde{X}] \\
&= \left( \prod_{j=1}^n |\tilde{a}_{jj}|^{2j} \right) [d\tilde{X}].
\end{aligned}$$

If the  $\tilde{x}_{jj}$ 's and  $\tilde{a}_{jj}$ 's are real then note that the  $x_{jk}$ 's for  $j > k$  contribute  $\tilde{a}_{jj}$  twice that is, corresponding to  $x_{jk}^{(1)}$  and  $x_{jk}^{(2)}$ , whereas the  $\tilde{a}_{jj}$ 's appear only once corresponding to the  $x_{jj}^{(1)}$ 's since the  $x_{jj}^{(2)}$ 's are zeros. This establishes (3.17). If  $\tilde{Y} = \tilde{X}\tilde{B}$  and if a matrix  $H_1$  is defined corresponding to  $G_1$  then note that the  $b_{jj}^{(1)}$ 's appear  $n - j + 1$  times on the diagonal for  $j = 1, \dots, n$ . Results (3.18) and (3.19) are established by using similar steps as in the case of (3.16) and (3.17).  $\square$

**Proposition 3.7.** *Let  $\tilde{X} \in \mathbb{C}^{n \times n}$  be hermitian matrix of independent complex entries and  $\tilde{A} \in \mathbb{C}^{n \times n}$  be a nonsingular matrix of constants. If  $\tilde{Y} = \tilde{A}\tilde{X}\tilde{A}^*$ , then*

$$[d\tilde{Y}] = \left| \det(\tilde{A}\tilde{A}^*) \right|^n [d\tilde{X}]. \quad (3.20)$$

*Proof.* Since  $\tilde{A}$  is nonsingular it can be written as a product of elementary matrices. Let  $\tilde{E}_1, \dots, \tilde{E}_k$  be elementary matrices such that

$$\tilde{A} = \tilde{E}_k \tilde{E}_{k-1} \cdots \tilde{E}_1 \implies \tilde{A}^* = \tilde{E}_1^* \tilde{E}_2^* \cdots \tilde{E}_k^*.$$

For example, let  $\tilde{E}_1$  be such that the  $j$ -th row of an identity matrix is multiplied by a scalar  $\tilde{c} = a + \sqrt{-1}b$  where  $a, b \in \mathbb{R}$ . Then  $\tilde{E}_1 \tilde{X} \tilde{E}_1^*$  means that the  $j$ -th row of  $\tilde{X}$  is multiplied by  $a + \sqrt{-1}b$  and the  $j$ -th column of  $\tilde{X}$  is multiplied by  $a - \sqrt{-1}b$ . Let

$$\tilde{U}_1 = \tilde{E}_1 \tilde{X} \tilde{E}_1^*, \quad \tilde{U}_2 = \tilde{E}_2 \tilde{U}_1 \tilde{E}_2^*, \quad \dots, \quad \tilde{U}_k = \tilde{E}_k \tilde{U}_{k-1} \tilde{E}_k^*.$$

Then the Jacobian of  $\tilde{Y}$  written as a function  $\tilde{X}$  is given by

$$J(\tilde{Y} : \tilde{X}) = J(\tilde{Y} : \tilde{U}_{k-1}) \cdots J(\tilde{U}_1 : \tilde{X}).$$

Let us evaluate  $[d\tilde{U}_1]$  in terms of  $[d\tilde{X}]$  by direct computation. Since  $\tilde{X}$  is hermitian its diagonal elements are real and the elements above the leading diagonal are the complex conjugates of those below the leading diagonal, and  $\tilde{U}_1$  is also of the same structure as  $\tilde{X}$ . Let  $\tilde{U}_1 = U + \sqrt{-1}V$  and  $\tilde{X} = Z + \sqrt{-1}W$  where  $U = [u_{jk}], V = [v_{jk}], Z = [z_{jk}], W = [w_{jk}]$  are all real and the diagonal elements of  $V$  and  $W$  are zeros. Take the  $u_{jj}$ 's and  $z_{jj}$ 's separately. The matrix of partial derivatives

of  $u_{11}, \dots, u_{nn}$  with respect to  $z_{11}, \dots, z_{nn}$  is a diagonal matrix with the  $j$ -th element  $a^2 + b^2$  and all other elements unities. That is,

$$\frac{\partial(\text{diag}(U))}{\partial(\text{diag}(Z))} = \frac{\partial(u_{11}, \dots, u_{nn})}{\partial(z_{11}, \dots, z_{nn})} = \begin{bmatrix} 1 & & & \\ & \ddots & & \\ & & a^2 + b^2 = |\tilde{c}|^2 & \\ & & & \ddots \\ & & & & 1 \end{bmatrix} := C,$$

where  $\text{diag}(X)$  means the diagonal matrix, obtained by keeping the diagonal entries of  $X$  and ignoring the off-diagonal entries.

The remaining variables produce a  $\frac{n(n-1)}{2} \times \frac{n(n-1)}{2}$  matrix of the following type

$$\frac{\partial(U_0, V_0)}{\partial(Z_0, W_0)} = \begin{bmatrix} A_0 & B_0 \\ -B_0 & A_0 \end{bmatrix}$$

where  $U_0, V_0, Z_0, W_0$  mean that the diagonal elements are deleted,  $A_0$  is a diagonal matrix with  $n - 1$  of the diagonal elements equal to  $a$  and the remaining unities and  $B_0$  is a diagonal matrix such that corresponding to every  $a$  in  $A_0$  there is a  $b$  or  $-b$  with  $j - 1$  of them equal to  $-b$  and  $n - j$  of them equal to  $b$ . Thus the Jacobian of this transformation is:

$$\begin{aligned} J(\tilde{U}_1 : \tilde{X}) &= \frac{\partial(\text{diag}(U), U_0, V_0)}{\partial(\text{diag}(Z), Z_0, W_0)} = \det \left( \begin{bmatrix} C & 0 & 0 \\ 0 & A_0 & B_0 \\ 0 & -B_0 & A_0 \end{bmatrix} \right) \\ &= \det(C) \det \left( \begin{bmatrix} A_0 & B_0 \\ -B_0 & A_0 \end{bmatrix} \right) \end{aligned}$$

From Lemma 3.1, the determinant is  $|\det(A_0 + \sqrt{-1}B_0)|^2$ . That is,

$$\begin{aligned} \det \left( \frac{\partial(U_0, V_0)}{\partial(Z_0, W_0)} \right) &= \det \left( \begin{bmatrix} A_0 & B_0 \\ -B_0 & A_0 \end{bmatrix} \right) \\ &= \left| \det((A_0 + \sqrt{-1}B_0)(A_0 + \sqrt{-1}B_0)^*) \right| \\ &= (a^2 + b^2)^{n-1}. \end{aligned}$$

Thus

$$[d\tilde{U}_1] = (a^2 + b^2)^n [d\tilde{X}] = \left| \det(\tilde{E}_1 \tilde{E}_1^*) \right|^n [d\tilde{X}].$$

Note that interchanges of rows and columns can produce only a change in the sign in the determinant, the addition of a row (column) to another row (column) does not change the determinant

and elementary matrices of the type  $\tilde{E}_1$  will produce  $\left| \det(\tilde{E}_1 \tilde{E}_1^*) \right|^n$  in the Jacobian. Thus by computing  $J(\tilde{U}_1 : \tilde{X}), J(\tilde{U}_2 : \tilde{U}_1)$  etc we have

$$[d\tilde{Y}] = \left| \det(\tilde{A} \tilde{A}^*) \right|^n [d\tilde{X}].$$

As a specific example, the configuration of the partial derivatives for  $n = 3$  with  $j = 2$  is the following: If

$$\tilde{X} = \begin{bmatrix} z_{11} & * & * \\ z_{21} + \sqrt{-1}w_{21} & z_{22} & * \\ z_{31} + \sqrt{-1}w_{31} & z_{32} + \sqrt{-1}w_{32} & z_{33} \end{bmatrix} \text{ and } \tilde{E}_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \tilde{c} & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

then

$$\begin{aligned} \tilde{U}_1 &= \begin{bmatrix} z_{11} & * & * \\ \tilde{c}(z_{21} + \sqrt{-1}w_{21}) & |\tilde{c}|^2 z_{22} & * \\ z_{31} + \sqrt{-1}w_{31} & \tilde{c}(z_{32} + \sqrt{-1}w_{32}) & z_{33} \end{bmatrix} \\ &= \begin{bmatrix} u_{11} & * & * \\ u_{21} + \sqrt{-1}v_{21} & u_{22} & * \\ u_{31} + \sqrt{-1}v_{31} & u_{32} + \sqrt{-1}v_{32} & u_{33} \end{bmatrix}, \end{aligned}$$

thus

$$\begin{aligned} u_{11} &= z_{11}, u_{22} = |\tilde{c}|^2 z_{22}, u_{33} = z_{33}, \\ u_{21} &= az_{21} - bw_{21}, u_{31} = z_{31}, u_{32} = az_{32} + bw_{32}, \\ v_{21} &= aw_{21} + bz_{21}, v_{31} = w_{31}, v_{32} = aw_{32} - bz_{32}. \end{aligned}$$

Now

$$\begin{bmatrix} u_{11} \\ u_{22} \\ u_{33} \\ u_{21} \\ u_{31} \\ u_{32} \\ v_{21} \\ v_{31} \\ v_{32} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & |\tilde{c}|^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a & 0 & 0 & -b & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & a & 0 & 0 & b \\ 0 & 0 & 0 & b & 0 & 0 & a & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -b & 0 & 0 & a \end{bmatrix} \begin{bmatrix} z_{11} \\ z_{22} \\ z_{33} \\ z_{21} \\ z_{31} \\ z_{32} \\ w_{21} \\ w_{31} \\ w_{32} \end{bmatrix}.$$

Now

$$C = \begin{bmatrix} 1 & & \\ & |\tilde{c}|^2 & \\ & & 1 \end{bmatrix}, A_0 = \begin{bmatrix} a & & \\ & 1 & \\ & & a \end{bmatrix}, B_0 = \begin{bmatrix} -b & & \\ & 0 & \\ & & b \end{bmatrix}.$$

We are done.  $\square$

**Remark 3.8.** If  $\tilde{X}$  is skew hermitian then the diagonal elements are purely imaginary, that is, the real parts are zeros. It is easy to note that the structure of the Jacobian matrix for a transformation of the type  $\tilde{Y} = \tilde{A}\tilde{X}\tilde{A}^*$ , where  $\tilde{X}^* = -\tilde{X}$ , remains the same as that in the hermitian case of Proposition 3.7. The roles of  $(u_{jj}, z_{jj})$ 's and  $(v_{jj}, w_{jj})$ 's are interchanged. Thus the next theorem will be stated without proof.

**Proposition 3.9.** Let  $\tilde{X} \in \mathbb{C}^{n \times n}$  skew hermitian matrix of independent complex entries. Let  $\tilde{A} \in \mathbb{C}^{n \times n}$  be a nonsingular matrix of constants. If  $\tilde{Y} = \tilde{A}\tilde{X}\tilde{A}^*$ , then

$$[\mathrm{d}\tilde{Y}] = \left| \det(\tilde{A}\tilde{A}^*) \right|^n [\mathrm{d}\tilde{X}]. \quad (3.21)$$

Some simple nonlinear transformations will be considered here. These are transformations which become linear transformations in the differentials so that the Jacobian of the original transformation becomes the Jacobian of the linear transformation where the matrices of differentials are treated as the new variables and everything else as constants.

**Proposition 3.10.** Let  $\tilde{X} \in \mathbb{C}^{n \times n}$  be hermitian positive definite matrix of independent complex variables. Let  $\tilde{T} \in \mathbb{C}^{n \times n}$  be lower triangular and  $\tilde{Q} \in \mathbb{C}^{n \times n}$  be upper triangular matrices of independent complex variables with real and positive diagonal elements. Then

$$\tilde{X} = \tilde{T}\tilde{T}^* \implies [\mathrm{d}\tilde{X}] = 2^n \left( \prod_{j=1}^n t_{jj}^{2(n-j)+1} \right) [\mathrm{d}\tilde{T}], \quad (3.22)$$

$$\tilde{X} = \tilde{Q}\tilde{Q}^* \implies [\mathrm{d}\tilde{X}] = 2^n \left( \prod_{j=1}^n t_{jj}^{2(j-1)+1} \right) [\mathrm{d}\tilde{Q}]. \quad (3.23)$$

*Proof.* When the diagonal elements of the triangular matrices are real and positive there exist unique representations  $\tilde{X} = \tilde{T}\tilde{T}^*$  and  $\tilde{X} = \tilde{Q}\tilde{Q}^*$ . Let  $\tilde{X} = X_1 + \sqrt{-1}X_2$  and  $\tilde{T} = T_1 + \sqrt{-1}T_2$ , where  $\tilde{X} = [\tilde{x}_{jk}]$ ,  $\tilde{T} = [\tilde{t}_{jk}]$ ,  $\tilde{t}_{jk} = 0, j < k$ ,  $X_m = [x_{jk}^{(m)}]$ ,  $T_m = [t_{jk}^{(m)}]$ ,  $m = 1, 2$ . Note that  $X_1$  is symmetric and  $X_2$  is skew symmetric. The diagonal elements of  $X_2$  and  $T_2$  are zeros. Hence when considering the Jacobian we should take  $\tilde{x}_{jj}, j = 1, \dots, p$  and  $\tilde{x}_{jk}, j > k$  separately.

$$\begin{aligned} \tilde{X} = \tilde{T}\tilde{T}^* &\implies X_1 + \sqrt{-1}X_2 = (T_1 + \sqrt{-1}T_2)(T_1^T - \sqrt{-1}T_2^T) \\ &\implies \begin{cases} X_1 = T_1 T_1^T + T_2 T_2^T \\ X_2 = T_2 T_1^T - T_1 T_2^T \end{cases}, \end{aligned}$$

with  $t_{jj}^{(1)} = t_{jj}$ ,  $t_{jj}^{(2)} = 0, j = 1, \dots, n$ . Note that

$$x_{jj}^{(1)} = \left( (t_{j1}^{(1)})^2 + \dots + (t_{jj}^{(1)})^2 \right) + \left( (t_{j1}^{(2)})^2 + \dots + (t_{j,j-1}^{(2)})^2 \right)$$

implies that

$$\frac{\partial x_{jj}^{(1)}}{\partial t_{jj}^{(1)}} = 2t_{jj}^{(1)} = 2t_{jj}, j = 1, \dots, n.$$

So

$$\frac{\partial(x_{11}^{(1)}, \dots, x_{nn}^{(1)})}{\partial(t_{11}^{(1)}, \dots, t_{nn}^{(1)})} = \begin{bmatrix} 2t_{11} & & \\ & \ddots & \\ & & 2t_{nn} \end{bmatrix} := Z.$$

Now consider the  $x_{jk}^{(1)}$ 's for  $j > k$ . It is easy to note that

$$\frac{\partial(X_{10}, X_{20})}{\partial(T_{10}, T_{20})} = \begin{bmatrix} U & V \\ W & Y \end{bmatrix},$$

where a zero indicates that the  $x_{jj}^{(1)}$ 's are removed and the derivatives are taken with respect to the  $t_{jk}^{(1)}$ 's and  $t_{jk}^{(2)}$ 's for  $j > k$ .  $U$  and  $Y$  are lower triangular matrices with  $t_{jj}$  repeated  $n - j$  times along the diagonal and  $V$  is of the same form as  $U$  but with  $t_{jj}^{(2)} = 0$  along the diagonal and the  $t_{jk}^{(1)}$ 's replaced by the  $t_{jk}^{(2)}$ 's. For example, take the  $x_{jk}^{(1)}$ 's in the order  $x_{21}^{(1)}, x_{31}^{(1)}, \dots, x_{n1}^{(1)}, x_{32}^{(1)}, \dots, x_{n,n-1}^{(1)}$  and  $t_{jk}^{(1)}$ 's also in the same order. Then we get the  $\frac{n(n-1)}{2} \times \frac{n(n-1)}{2}$  matrix

$$U = \frac{\partial X_{10}}{\partial T_{10}} = \begin{bmatrix} t_{11} & 0 & \cdots & 0 \\ * & t_{11} & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ * & * & \cdots & t_{n-1,n-1} \end{bmatrix}$$

where the  $*$ 's indicate the presence of elements some of which may be zeros. Since  $U$  and  $V$  are lower triangular with the diagonal elements of  $V$  being zeros, one can make  $W$  null by adding suitable combinations of the rows of  $(U, V)$ . This will not alter the lower triangular nature or the diagonal elements of  $Y$ . Then the determinant is given by

$$\det \left( \begin{bmatrix} U & V \\ W & Y \end{bmatrix} \right) = \det(U) \det(Y) = \prod_{j=1}^n t_{jj}^{2(n-j)}.$$

Multiply with the  $2t_{jj}$ 's for  $j = 1, \dots, n$  to establish the result. As a specific example, we consider the case where  $n = 3$ . Let  $\tilde{X} \in \mathbb{C}^{3 \times 3}$ . Denote  $\tilde{X} = X_1 + \sqrt{-1}X_2$ . Thus

$$X_1 = \begin{bmatrix} x_{11}^{(1)} & x_{21}^{(1)} & x_{31}^{(1)} \\ x_{21}^{(1)} & x_{22}^{(1)} & x_{32}^{(1)} \\ x_{31}^{(1)} & x_{32}^{(1)} & x_{33}^{(1)} \end{bmatrix}, \quad X_2 = \begin{bmatrix} 0 & -x_{21}^{(2)} & -x_{31}^{(2)} \\ x_{21}^{(2)} & 0 & -x_{32}^{(2)} \\ x_{31}^{(2)} & x_{32}^{(2)} & 0 \end{bmatrix}.$$

Similarly, let  $\tilde{T} = T_1 + \sqrt{-1}T_2$ . We also have:

$$T_1 = \begin{bmatrix} t_{11}^{(1)} & 0 & 0 \\ t_{21}^{(1)} & t_{22}^{(1)} & 0 \\ t_{31}^{(1)} & t_{32}^{(1)} & t_{33}^{(1)} \end{bmatrix}, \quad T_2 = \begin{bmatrix} 0 & 0 & 0 \\ t_{21}^{(2)} & 0 & 0 \\ t_{31}^{(2)} & t_{32}^{(2)} & 0 \end{bmatrix}.$$

Now  $X_1 = T_1 T_1^\top + T_2 T_2^\top$  can be expanded as follows:

$$\begin{aligned} x_{11}^{(1)} &= \left(t_{11}^{(1)}\right)^2, \quad x_{21}^{(1)} = t_{21}^{(1)} t_{11}^{(1)}, \quad x_{31}^{(1)} = t_{31}^{(1)} t_{11}^{(1)}; \\ x_{22}^{(1)} &= \left(t_{21}^{(1)}\right)^2 + \left(t_{22}^{(1)}\right)^2 + \left(t_{21}^{(2)}\right)^2, \quad x_{32}^{(1)} = t_{31}^{(1)} t_{21}^{(1)} + t_{32}^{(1)} t_{22}^{(1)} + t_{31}^{(2)} t_{21}^{(2)}; \\ x_{33}^{(1)} &= \left(t_{31}^{(1)}\right)^2 + \left(t_{32}^{(1)}\right)^2 + \left(t_{33}^{(1)}\right)^2 + \left(t_{31}^{(2)}\right)^2 + \left(t_{32}^{(2)}\right)^2. \end{aligned}$$

Then  $X_2 = T_2 T_1^\top - T_1 T_2^\top$  can be expanded as follows:

$$x_{21}^{(2)} = t_{21}^{(2)} t_{11}^{(1)}, \quad x_{31}^{(2)} = t_{31}^{(2)} t_{11}^{(1)}, \quad x_{32}^{(2)} = t_{31}^{(2)} t_{21}^{(1)} + t_{32}^{(2)} t_{22}^{(1)}.$$

From the above, we see that

$$\begin{bmatrix} dx_{11}^{(1)} \\ dx_{22}^{(1)} \\ dx_{33}^{(1)} \\ dx_{21}^{(1)} \\ dx_{31}^{(1)} \\ dx_{32}^{(1)} \\ dx_{21}^{(2)} \\ dx_{31}^{(2)} \\ dx_{32}^{(2)} \end{bmatrix} = \begin{bmatrix} 2t_{11}^{(1)} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2t_{22}^{(1)} & 0 & 2t_{21}^{(1)} & 0 & 0 & 2t_{21}^{(2)} & 0 & 0 \\ 0 & 0 & 2t_{33}^{(1)} & 0 & 2t_{31}^{(1)} & 2t_{32}^{(1)} & 0 & 2t_{31}^{(2)} & 2t_{32}^{(2)} \\ t_{21}^{(1)} & 0 & 0 & t_{11}^{(1)} & 0 & 0 & 0 & 0 & 0 \\ t_{31}^{(1)} & 0 & 0 & 0 & t_{11}^{(1)} & 0 & 0 & 0 & 0 \\ 0 & t_{32}^{(1)} & 0 & t_{31}^{(1)} & t_{21}^{(1)} & t_{22}^{(1)} & t_{31}^{(2)} & t_{21}^{(2)} & 0 \\ t_{21}^{(2)} & 0 & 0 & 0 & 0 & 0 & t_{11}^{(1)} & 0 & 0 \\ t_{31}^{(2)} & 0 & 0 & 0 & 0 & 0 & 0 & t_{11}^{(1)} & 0 \\ 0 & t_{32}^{(2)} & 0 & t_{31}^{(2)} & 0 & 0 & 0 & t_{21}^{(1)} & t_{22}^{(1)} \end{bmatrix} \begin{bmatrix} dt_{11}^{(1)} \\ dt_{22}^{(1)} \\ dt_{33}^{(1)} \\ dt_{21}^{(1)} \\ dt_{31}^{(1)} \\ dt_{32}^{(1)} \\ dt_{21}^{(2)} \\ dt_{31}^{(2)} \\ dt_{32}^{(2)} \end{bmatrix}.$$

In what follows, we compute the Jacobian of this transformation:

$$\begin{aligned}
J(\tilde{X} : \tilde{T}) &= \det \left( \begin{bmatrix} 2t_{11}^{(1)} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2t_{22}^{(1)} & 0 & 2t_{21}^{(1)} & 0 & 0 & 2t_{21}^{(2)} & 0 & 0 \\ 0 & 0 & 2t_{33}^{(1)} & 0 & 2t_{31}^{(1)} & 2t_{32}^{(1)} & 0 & 2t_{31}^{(2)} & 2t_{32}^{(2)} \\ t_{21}^{(1)} & 0 & 0 & t_{11}^{(1)} & 0 & 0 & 0 & 0 & 0 \\ t_{31}^{(1)} & 0 & 0 & 0 & t_{11}^{(1)} & 0 & 0 & 0 & 0 \\ 0 & t_{32}^{(1)} & 0 & t_{31}^{(1)} & t_{21}^{(1)} & t_{22}^{(1)} & t_{31}^{(2)} & t_{21}^{(2)} & 0 \\ t_{21}^{(2)} & 0 & 0 & 0 & 0 & 0 & t_{11}^{(1)} & 0 & 0 \\ t_{31}^{(2)} & 0 & 0 & 0 & 0 & 0 & 0 & t_{11}^{(1)} & 0 \\ 0 & t_{32}^{(2)} & 0 & t_{31}^{(2)} & 0 & 0 & 0 & t_{21}^{(1)} & t_{22}^{(1)} \end{bmatrix} \right) \\
&= 2t_{11}^{(1)} \det \left( \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2t_{22}^{(1)} & 0 & 2t_{21}^{(1)} & 0 & 0 & 2t_{21}^{(2)} & 0 & 0 \\ 0 & 0 & 2t_{33}^{(1)} & 0 & 2t_{31}^{(1)} & 2t_{32}^{(1)} & 0 & 2t_{31}^{(2)} & 2t_{32}^{(2)} \\ t_{21}^{(1)} & 0 & 0 & t_{11}^{(1)} & 0 & 0 & 0 & 0 & 0 \\ t_{31}^{(1)} & 0 & 0 & 0 & t_{11}^{(1)} & 0 & 0 & 0 & 0 \\ 0 & t_{32}^{(1)} & 0 & t_{31}^{(1)} & t_{21}^{(1)} & t_{22}^{(1)} & t_{31}^{(2)} & t_{21}^{(2)} & 0 \\ t_{21}^{(2)} & 0 & 0 & 0 & 0 & 0 & t_{11}^{(1)} & 0 & 0 \\ t_{31}^{(2)} & 0 & 0 & 0 & 0 & 0 & 0 & t_{11}^{(1)} & 0 \\ 0 & t_{32}^{(2)} & 0 & t_{31}^{(2)} & 0 & 0 & 0 & t_{21}^{(1)} & t_{22}^{(1)} \end{bmatrix} \right),
\end{aligned}$$

by adding the corresponding multiples of the first row to the second row through the last one, respectively, we get

$$J(\tilde{X} : \tilde{T}) = 2t_{11}^{(1)} \det \left( \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2t_{22}^{(1)} & 0 & 2t_{21}^{(1)} & 0 & 0 & 2t_{21}^{(2)} & 0 & 0 \\ 0 & 0 & 2t_{33}^{(1)} & 0 & 2t_{31}^{(1)} & 2t_{32}^{(1)} & 0 & 2t_{31}^{(2)} & 2t_{32}^{(2)} \\ 0 & 0 & 0 & t_{11}^{(1)} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & t_{11}^{(1)} & 0 & 0 & 0 & 0 \\ 0 & t_{32}^{(1)} & 0 & t_{31}^{(1)} & t_{21}^{(1)} & t_{22}^{(1)} & t_{31}^{(2)} & t_{21}^{(2)} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & t_{11}^{(1)} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & t_{11}^{(1)} & 0 \\ 0 & t_{32}^{(2)} & 0 & t_{31}^{(2)} & 0 & 0 & 0 & t_{21}^{(1)} & t_{22}^{(1)} \end{bmatrix} \right),$$

iteratively, finally we get the final result. We also take a simple approach (i.e. by definition) with

a tedious computation as follows:

$$\begin{aligned}
dx_{11}^{(1)} &= 2t_{11}dt_{11}, \quad dx_{21}^{(1)} = t_{21}^{(1)}dt_{11} + t_{11}dt_{21}^{(1)}, \quad dx_{31}^{(1)} = t_{31}^{(1)}dt_{11} + t_{11}dt_{31}^{(1)}; \\
dx_{22}^{(1)} &= 2t_{22}dt_{22} + 2t_{21}^{(1)}dt_{21}^{(1)} + 2t_{21}^{(2)}dt_{21}^{(2)}, \\
dx_{32}^{(1)} &= t_{31}^{(1)}dt_{21}^{(1)} + t_{21}^{(1)}dt_{31}^{(1)} + t_{32}^{(1)}dt_{22} + t_{22}dt_{32}^{(1)} + t_{31}^{(2)}dt_{21}^{(2)} + t_{21}^{(2)}dt_{31}^{(2)}; \\
dx_{33}^{(1)} &= 2t_{31}^{(1)}dt_{31}^{(1)} + 2t_{32}^{(1)}dt_{32}^{(1)} + 2t_{33}dt_{33} + 2t_{31}^{(2)}dt_{31}^{(2)} + 2t_{32}^{(2)}dt_{32}^{(2)},
\end{aligned}$$

and

$$\begin{aligned}
dx_{21}^{(2)} &= t_{21}^{(2)}dt_{11} + t_{11}dt_{21}^{(2)}, \quad dx_{31}^{(2)} = t_{31}^{(2)}dt_{11} + t_{11}dt_{31}^{(2)}, \\
dx_{32}^{(2)} &= t_{31}^{(2)}dt_{21}^{(1)} + t_{21}^{(1)}dt_{31}^{(2)} + t_{22}dt_{32}^{(2)} + t_{32}^{(2)}dt_{22}.
\end{aligned}$$

Hence we can compute the Jacobian by definition as follows:

$$\begin{aligned}
[d\tilde{X}] &= dx_{11}^{(1)} \wedge dx_{21}^{(1)} \wedge dx_{31}^{(1)} \wedge dx_{21}^{(2)} \wedge dx_{31}^{(2)} \wedge dx_{22}^{(1)} \wedge dx_{32}^{(2)} \wedge dx_{32}^{(1)} \wedge dx_{33}^{(1)} \\
&= 2^3 \left( \prod_{j=1}^3 t_{jj}^{2(n-j)+1} \right) [d\tilde{T}].
\end{aligned}$$

The proof in the case of  $\tilde{X} = \tilde{Q}\tilde{Q}^*$  is similar but in this case it can be seen that the triangular matrices corresponding to  $U$  and  $Y$  will have  $t_{jj}$  repeated  $j-1$  times along the diagonal for  $j = 1, \dots, n$ .  $\square$

**Example 3.11.** If  $\tilde{X}^* = \tilde{X} \in \mathbb{C}^{n \times n}$  is positive definite, and  $\text{Re}(\alpha) > n-1$ ,

$$\begin{aligned}
\tilde{\Gamma}_n(\alpha) &:= \int_{\tilde{X} > 0} [d\tilde{X}] \left| \det(\tilde{X}) \right|^{\alpha-n} e^{-\text{Tr}(\tilde{X})} \\
&= \pi^{\frac{n(n-1)}{2}} \Gamma(\alpha) \Gamma(\alpha-1) \cdots \Gamma(\alpha-n+1).
\end{aligned} \tag{3.24}$$

Indeed, let  $\tilde{T} = [\tilde{t}_{jk}]$ ,  $\tilde{t}_{jk} = 0, j < k$  be a lower triangular matrix with real and positive diagonal elements  $t_{jj} > 0, j = 1, \dots, n$  such that  $\tilde{X} = \tilde{T}\tilde{T}^*$ . Then from Proposition 3.10

$$[d\tilde{X}] = 2^n \left( \prod_{j=1}^n t_{jj}^{2(n-j)+1} \right) [d\tilde{T}]$$

Note that

$$\text{Tr}(\tilde{X}) = \text{Tr}(\tilde{T}\tilde{T}^*) = \sum_{j=1}^n t_{jj}^2 + |\tilde{t}_{21}|^2 + \cdots + |\tilde{t}_{n1}|^2 + \cdots + |\tilde{t}_{n,n-1}|^2$$

and

$$\left| \det(\tilde{X}) \right|^{\alpha-n} [d\tilde{X}] = 2^n \left( \prod_{j=1}^n t_{jj}^{2\alpha-2j+1} \right) [d\tilde{T}].$$

The integral over  $\tilde{X}$  splits into  $n$  integrals over the  $t_{jj}$ 's and  $\frac{n(n-1)}{2}$  integrals over the  $\tilde{t}_{jk}$ 's,  $j > k$ . Note that  $0 < t_{jj} < \infty$ ,  $-\infty < t_{jk}^{(1)} < \infty$ ,  $-\infty < t_{jk}^{(2)} < \infty$ , where  $\tilde{t}_{jk} = t_{jk}^{(1)} + \sqrt{-1}t_{jk}^{(2)}$ . But

$$2 \int_0^\infty t_{jj}^{2\alpha-2j+1} e^{-t_{jj}^2} dt_{jj} = \Gamma(\alpha - j + 1), \quad \text{Re}(\alpha) > j - 1,$$

for  $j = 1, \dots, n$ , so  $\text{Re}(\alpha) > n - 1$  and

$$\int_{\tilde{t}_{jk}} e^{-|\tilde{t}_{jk}|^2} d\tilde{t}_{jk} = \int_{-\infty}^\infty \int_{-\infty}^\infty e^{-\left((t_{jk}^{(1)})^2 + (t_{jk}^{(2)})^2\right)} dt_{jk}^{(1)} dt_{jk}^{(2)} = \pi.$$

The desired result is obtained.

**Definition 3.12** ( $\tilde{\Gamma}_n(\alpha)$ : complex matrix-variate gamma). It is defined as stated in Example 3.11. We will write with a *tilda* over  $\Gamma$  to distinguish it from the matrix-variate gamma in the real case.

**Example 3.13.** Show that

$$f(\tilde{X}) = \frac{|\det(\tilde{B})|^\alpha |\det(\tilde{X})|^{\alpha-n} e^{-\text{Tr}(\tilde{B}\tilde{X})}}{\tilde{\Gamma}_n(\alpha)}$$

for  $\tilde{B}^* = \tilde{B} > 0$ ,  $\tilde{X}^* = \tilde{X} > 0$ ,  $\text{Re}(\alpha) > n - 1$  and  $f(\tilde{X}) = 0$  elsewhere, is a density function for  $\tilde{X}$  where  $\tilde{B}$  is a constant matrix, with  $\tilde{\Gamma}_n(\alpha)$  as given in Definition 3.12. Indeed, evidently  $f(\tilde{X}) \geq 0$  for all  $\tilde{X}$  and for all  $\tilde{X}$  and it remains to show that the total integral is unity. Since  $\tilde{B}$  is hermitian positive definite there exists a nonsingular  $\tilde{C}$  such that  $\tilde{B} = \tilde{C}^* \tilde{C}$ . Then

$$\text{Tr}(\tilde{B}\tilde{X}) = \text{Tr}(\tilde{C}\tilde{X}\tilde{C}^*).$$

Hence from Proposition 3.10

$$\tilde{Y} = \tilde{C}\tilde{X}\tilde{C}^* \implies [d\tilde{Y}] = |\det(\tilde{C}\tilde{C}^*)|^n [d\tilde{X}] = |\det(\tilde{B})|^n [d\tilde{X}],$$

and

$$\tilde{X} = \tilde{C}^{-1}\tilde{Y}\tilde{C}^{*-1} \implies |\det(\tilde{X})| = |\det(\tilde{C}\tilde{C}^*)|^{-1} |\det(\tilde{Y})|$$

Then

$$\int_{\tilde{X}>0} f(\tilde{X}) [d\tilde{X}] = \int_{\tilde{Y}>0} [d\tilde{Y}] \frac{|\det(\tilde{Y})|^{\alpha-n} e^{-\text{Tr}(\tilde{Y})}}{\tilde{\Gamma}_n(\alpha)} = 1.$$

But from Example 3.11, the right side is unity for  $\text{Re}(\alpha) > n - 1$ . This density  $f(\tilde{X})$  is known as the *complex matrix-variate density with the parameters  $\alpha$  and  $\tilde{B}$* .

### 3.2 The computation of volumes

**Definition 3.14** (Semiunitarity and unitary matrices). A  $p \times n$  matrix  $\tilde{U}$  is said to be *semiunitary* if  $\tilde{U}\tilde{U}^* = \mathbb{1}_p$  for  $p < n$  or  $\tilde{U}^*\tilde{U} = \mathbb{1}_n$  for  $p > n$ . When  $n = p$  and  $\tilde{U}\tilde{U}^* = \mathbb{1}_n$ , then  $\tilde{U}$  is called a *unitary matrix*. The set of all  $n \times n$  unitary matrices is denoted by  $\mathcal{U}(n)$ . That is,

$$\boxed{\mathcal{U}(n) := \left\{ \tilde{U} \in \mathbb{C}^{n \times n} : \tilde{U}\tilde{U}^* = \mathbb{1}_n \right\}}, \quad (3.25)$$

where  $\mathbb{C}^{n \times n}$  denotes the set of all  $n \times n$  complex matrices.

**Definition 3.15** (A hermitian or a skew hermitian matrix). Let  $\tilde{A} \in \mathbb{C}^{n \times n}$ . If  $\tilde{A} = \tilde{A}^*$ , then  $\tilde{A}$  is said to be *hermitian* and if  $\tilde{A}^* = -\tilde{A}$ , then it is *skew hermitian*.

When dealing with unitary matrices a basic property to be noted is the following:

$$\tilde{U}\tilde{U}^* = \mathbb{1} \implies \tilde{U}^* d\tilde{U} = -d\tilde{U}^* \tilde{U}.$$

But  $(\tilde{U}^* d\tilde{U})^* = d\tilde{U}^* \tilde{U}$ , which means that  $\tilde{U}^* d\tilde{U}$  is a skew hermitian matrix. The wedge product of  $\tilde{U}^* d\tilde{U}$ , namely,  $\wedge (\tilde{U}^* d\tilde{U})$  enters into the picture when evaluating the Jacobians involving unitary transformations. Hence this will be denoted by  $d\tilde{G}$  for convenience. Starting from  $\tilde{U}^* \tilde{U} = \mathbb{1}_n$  one has  $d\tilde{U} \cdot \tilde{U}^*$ .

Assume that  $\tilde{U} = [\tilde{u}_{ij}] \in \mathcal{U}(n)$  where  $\tilde{u}_{ij} \in \mathbb{C}$ . Let  $\tilde{u}_{jj} = |\tilde{u}_{jj}| e^{\sqrt{-1}\theta_j}$  by Euler's formula, where  $\theta_j \in [-\pi, \pi]$ . Then

$$\tilde{U} = \begin{bmatrix} |\tilde{u}_{11}| & \tilde{u}_{12}e^{-\sqrt{-1}\theta_2} & \dots & \tilde{u}_{1n}e^{-\sqrt{-1}\theta_n} \\ \tilde{u}_{21}e^{-\sqrt{-1}\theta_1} & |\tilde{u}_{22}| & \dots & \tilde{u}_{2n}e^{-\sqrt{-1}\theta_n} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{u}_{n1}e^{-\sqrt{-1}\theta_1} & \tilde{u}_{n2}e^{-\sqrt{-1}\theta_2} & \dots & |\tilde{u}_{nn}| \end{bmatrix} \begin{bmatrix} e^{\sqrt{-1}\theta_1} & 0 & \dots & 0 \\ 0 & e^{\sqrt{-1}\theta_2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & e^{\sqrt{-1}\theta_n} \end{bmatrix}. \quad (3.26)$$

This indicates that any  $\tilde{U} \in \mathcal{U}(n)$  can be factorized into a product of a unitary matrix with diagonal entries being nonnegative and a diagonal unitary matrix. It is easily seen that such factorization of a given unitary matrix is unique. In fact, we have a correspondence which is one-to-one:

$$\boxed{\mathcal{U}(n) \sim (\mathcal{U}(n)/\mathcal{U}(1)^{\times n}) \times \mathcal{U}(1)^{\times n}}.$$

**Notation.** When  $\tilde{U}$  is a  $n \times n$  unitary matrix of independent complex entries,  $\tilde{U}^*$  its conjugate transpose and  $d\tilde{U}$  the matrix of differentials then the wedge product in  $d\tilde{G} := d\tilde{U} \cdot \tilde{U}^*$  will be denoted by  $[d\tilde{G}]$ . That is, ignoring the sign,

$$\boxed{[d\tilde{G}] := \wedge (d\tilde{U} \cdot \tilde{U}^*) = \wedge (\tilde{U} \cdot d\tilde{U}^*)}.$$

If the diagonal entries or the entries in one row of this unitary matrix  $\tilde{U}$  are assumed to be real, then the skew hermitian matrix  $d\tilde{U} \cdot \tilde{U}^*$  will be denoted by  $d\tilde{G}_1$  and its wedge product by

$$[d\tilde{G}_1] := \wedge (d\tilde{U} \cdot \tilde{U}^*).$$

Indeed,  $[d\tilde{G}]$  here means the wedge product over  $\mathcal{U}(n)$ , but however  $[d\tilde{G}_1]$  means the wedge product over  $\mathcal{U}(n)/\mathcal{U}(1)^{\times n}$ . Therefore, for any measurable function  $f$  over  $\mathcal{U}(n)$ ,  $[d\tilde{G}] = [d\tilde{G}_1][d\tilde{D}]$ ,

$$\int_{\mathcal{U}(n)} f(\tilde{U}) [d\tilde{G}] = \int_{\mathcal{U}_1(n)} \int_{\mathcal{U}(1)^{\times n}} f(\tilde{V}\tilde{D}) [d\tilde{G}_1][d\tilde{D}],$$

where  $\tilde{U} = \tilde{V}\tilde{D}$  for  $\tilde{V} \in \mathcal{U}_1(n)$ ,  $d\tilde{G} = \tilde{U}^* d\tilde{U}$  and  $d\tilde{G}_1 = \tilde{V}^* d\tilde{V}$ . Furthermore, let  $\tilde{V} = [\tilde{v}_{ij}]$  for  $\tilde{v}_{ij} \in \mathbb{C}$  and  $\tilde{v}_{jj} = v_{jj} \in \mathbb{R}^+$ . Then it holds still that  $\tilde{V} \in \mathcal{U}(n)$ , thus  $v_{jj}$  is not an independent variable, for example,  $v_{11} = \sqrt{1 - |\tilde{v}_{21}|^2 - \dots - |\tilde{v}_{n1}|^2}$ . From this, we see that

$$[d\tilde{G}_1] = \prod_{i < j} d \left( \operatorname{Re}(\tilde{V}^* d\tilde{V})_{ij} \right) d \left( \operatorname{Im}(\tilde{V}^* d\tilde{V})_{ij} \right)$$

and

$$[d\tilde{G}] = \left( \prod_{j=1}^n \operatorname{Im}(\tilde{U}^* d\tilde{U})_{jj} \right) \times \prod_{i < j} \operatorname{Re}(\tilde{U}^* d\tilde{U})_{ij} \operatorname{Im}(\tilde{U}^* d\tilde{U})_{ij}.$$

**Proposition 3.16.** *Let  $\tilde{T} \in \mathbb{C}^{n \times n}$  be lower triangular and  $\tilde{U} \in \mathcal{U}(n)$  be of independent complex variables. Let  $\tilde{X} = \tilde{T}\tilde{U}$ . Then:*

(i) *for all the diagonal entries  $t_{jj}, j = 1, \dots, n$  of  $\tilde{T}$  being real and positive,*

$$\boxed{[d\tilde{X}] = \left( \prod_{j=1}^n t_{jj}^{2(n-j)+1} \right) [d\tilde{T}][d\tilde{G}]} \quad (3.27)$$

*where  $d\tilde{G} = d\tilde{U} \cdot \tilde{U}^*$ ; and*

(ii) *for all the diagonal entries in  $\tilde{U}$  being real,*

$$[d\tilde{X}] = \left( \prod_{j=1}^n |\tilde{t}_{jj}|^{2(n-j)} \right) \cdot [d\tilde{T}][d\tilde{G}_1] \quad (3.28)$$

*where  $d\tilde{G}_1 = d\tilde{U} \cdot \tilde{U}^*$ .*

*Proof.* Taking differentials in  $\tilde{X} = \tilde{T}\tilde{U}$  one has

$$d\tilde{X} = d\tilde{T} \cdot \tilde{U} + \tilde{T} \cdot d\tilde{U}.$$

Postmultiplying by  $\tilde{U}^*$  and observing that  $\tilde{U}\tilde{U}^* = \mathbb{1}_n$  we have

$$d\tilde{X} \cdot \tilde{U}^* = d\tilde{T} + \tilde{T} \cdot d\tilde{U} \cdot \tilde{U}^*. \quad (3.29)$$

(i). Let the diagonal elements in  $\tilde{T}$  be real and positive and all other elements in  $\tilde{T}$  and  $\tilde{U}$  be complex. Let

$$d\tilde{V} = d\tilde{X} \cdot \tilde{U}^* \implies [d\tilde{V}] = [d\tilde{X}] \quad (3.30)$$

ignoring the sign, since  $\tilde{U}$  is unitary. Let  $d\tilde{G} = d\tilde{U} \cdot \tilde{U}^*$  and its wedge product be  $[d\tilde{G}]$ . Then

$$d\tilde{V} = d\tilde{T} + \tilde{T} \cdot d\tilde{G} \quad (3.31)$$

where  $d\tilde{G}$  is skew hermitian. Write

$$\begin{aligned} \tilde{V} &= [\tilde{v}_{jk}], \quad \tilde{v}_{jk} = v_{jk}^{(1)} + \sqrt{-1}v_{jk}^{(2)}, \\ \tilde{T} &= [\tilde{t}_{jk}], \quad \tilde{t}_{jk} = t_{jk}^{(1)} + \sqrt{-1}t_{jk}^{(2)}, j > k, \quad t_{jj}^{(1)} = t_{jj} > 0, \quad t_{jj}^{(2)} = 0, \\ d\tilde{G} &= [d\tilde{g}_{jk}], \quad d\tilde{g}_{jk} = dg_{jk}^{(1)} + \sqrt{-1}dg_{jk}^{(2)}, \quad dg_{jk}^{(1)} = -dg_{kj}^{(1)}, \quad dg_{jk}^{(2)} = dg_{kj}^{(2)}. \end{aligned}$$

From Eq. (3.31),

$$d\tilde{v}_{jk} = \begin{cases} d\tilde{t}_{jk} + (\tilde{t}_{j1}d\tilde{g}_{1k} + \dots + \tilde{t}_{jj}d\tilde{g}_{jk}), & j \geq k \\ (\tilde{t}_{j1}d\tilde{g}_{1k} + \dots + \tilde{t}_{jj}d\tilde{g}_{jk}), & j < k. \end{cases}$$

The new variables are  $dv_{jk}^{(m)} = \hat{v}_{jk}^{(m)}, dt_{jk}^{(m)} = \hat{t}_{jk}^{(m)}, dg_{jk}^{(m)} = \hat{g}_{jk}^{(m)}, j \geq k, m = 1, 2$ . The matrices of partial derivatives are easily seen to be the following:

$$\begin{aligned} \left[ \frac{\partial \hat{v}_{jj}^{(1)}}{\partial \hat{t}_{jj}} \right] &= \mathbb{1}, \quad \left[ \frac{\partial \hat{v}_{jk}^{(m)}}{\partial \hat{t}_{jk}^{(m)}}, j > k \right] = \mathbb{1}, \quad m = 1, 2, \\ \left[ \frac{\partial \hat{v}_{kj}^{(1)}}{\partial \hat{g}_{kj}^{(1)}}, j > k \right] &= A, \quad \left[ \frac{\partial \hat{v}_{kj}^{(2)}}{\partial \hat{g}_{kj}^{(2)}}, j > k \right] = B \end{aligned}$$

where  $A$  and  $B$  are triangular matrices with  $t_{jj}$  repeated  $n - j$  times,

$$\left[ \frac{\partial \hat{v}_{jj}^{(2)}}{\partial \hat{g}_{jj}^{(2)}} \right] = \text{diag}(t_{11}, \dots, t_{nn}).$$

By using the above identity matrices one can wipe out other submatrices in the same rows and columns and using the triangular blocks one can wipe out other blocks below it when evaluating the determinant of the Jacobian matrix and finally the determinant in absolute value reduces to the form

$$\det(A) \det(B) t_{11} \cdots t_{nn} = \prod_{j=1}^n t_{jj}^{2(n-j)+1}.$$

Hence the result. As a specific example, we consider the case where  $n = 3$ . We expand the expression:  $d\tilde{V} = d\tilde{T} + \tilde{T} \cdot d\tilde{G}$ . That is,

$$\begin{aligned} dV_1 &= dT_1 + T_1 d(\operatorname{Re}(\tilde{G})) - T_2 d(\operatorname{Im}(\tilde{G})), \\ dV_2 &= dT_2 + T_2 d(\operatorname{Re}(\tilde{G})) + T_1 d(\operatorname{Im}(\tilde{G})). \end{aligned}$$

Furthermore,

$$\begin{aligned} & \begin{bmatrix} dv_{11}^{(1)} & dv_{12}^{(1)} & dv_{13}^{(1)} \\ dv_{21}^{(1)} & dv_{22}^{(1)} & dv_{23}^{(1)} \\ dv_{31}^{(1)} & dv_{32}^{(1)} & dv_{33}^{(1)} \end{bmatrix} \\ &= \begin{bmatrix} dt_{11}^{(1)} & 0 & 0 \\ dt_{21}^{(1)} & dt_{22}^{(1)} & 0 \\ dt_{31}^{(1)} & dt_{32}^{(1)} & dt_{33}^{(1)} \end{bmatrix} + \begin{bmatrix} t_{11}^{(1)} & 0 & 0 \\ t_{21}^{(1)} & t_{22}^{(1)} & 0 \\ t_{31}^{(1)} & t_{32}^{(1)} & t_{33}^{(1)} \end{bmatrix} \begin{bmatrix} 0 & dg_{12}^{(1)} & dg_{13}^{(1)} \\ -dg_{12}^{(1)} & 0 & dg_{23}^{(1)} \\ -dg_{13}^{(1)} & -dg_{23}^{(1)} & 0 \end{bmatrix} \\ &\quad - \begin{bmatrix} 0 & 0 & 0 \\ t_{21}^{(2)} & 0 & 0 \\ t_{31}^{(2)} & t_{32}^{(2)} & 0 \end{bmatrix} \begin{bmatrix} dg_{11}^{(2)} & dg_{12}^{(2)} & dg_{13}^{(2)} \\ dg_{12}^{(2)} & dg_{22}^{(2)} & dg_{23}^{(2)} \\ dg_{13}^{(2)} & dg_{23}^{(2)} & dg_{33}^{(2)} \end{bmatrix} \end{aligned}$$

and

$$\begin{aligned} & \begin{bmatrix} dv_{11}^{(2)} & dv_{12}^{(2)} & dv_{13}^{(2)} \\ dv_{21}^{(2)} & dv_{22}^{(2)} & dv_{23}^{(2)} \\ dv_{31}^{(2)} & dv_{32}^{(2)} & dv_{33}^{(2)} \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 & 0 \\ dt_{21}^{(2)} & 0 & 0 \\ dt_{31}^{(2)} & dt_{32}^{(2)} & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ t_{21}^{(2)} & 0 & 0 \\ t_{31}^{(2)} & t_{32}^{(2)} & 0 \end{bmatrix} \begin{bmatrix} 0 & dg_{12}^{(1)} & dg_{13}^{(1)} \\ -dg_{12}^{(1)} & 0 & dg_{23}^{(1)} \\ -dg_{13}^{(1)} & -dg_{23}^{(1)} & 0 \end{bmatrix} \\ &\quad + \begin{bmatrix} t_{11}^{(1)} & 0 & 0 \\ t_{21}^{(1)} & t_{22}^{(1)} & 0 \\ t_{31}^{(1)} & t_{32}^{(1)} & t_{33}^{(1)} \end{bmatrix} \begin{bmatrix} dg_{11}^{(2)} & dg_{12}^{(2)} & dg_{13}^{(2)} \\ dg_{12}^{(2)} & dg_{22}^{(2)} & dg_{23}^{(2)} \\ dg_{13}^{(2)} & dg_{23}^{(2)} & dg_{33}^{(2)} \end{bmatrix}. \end{aligned}$$

Thus

$$\begin{aligned} dv_{11}^{(1)} &= dt_{11}, \quad dv_{12}^{(1)} = t_{11} dg_{12}^{(1)}, \quad dv_{13}^{(1)} = t_{11} dg_{13}^{(1)}, \\ dv_{21}^{(1)} &= dt_{21} - t_{22} dg_{12}^{(1)} - t_{21}^{(2)} dg_{11}^{(2)}, \quad dv_{22}^{(1)} = dt_{22} + t_{21}^{(1)} dg_{12}^{(1)} - t_{21}^{(2)} dg_{12}^{(2)}, \\ dv_{23}^{(1)} &= t_{21}^{(1)} dg_{13}^{(1)} + t_{22} dg_{23}^{(1)} - t_{21}^{(2)} dg_{13}^{(2)}, \\ dv_{31}^{(1)} &= dt_{31} - t_{32}^{(1)} dg_{12}^{(1)} - t_{33} dg_{13}^{(1)} - t_{31}^{(2)} dg_{11}^{(2)} - t_{32}^{(2)} dg_{12}^{(2)}, \\ dv_{32}^{(1)} &= dt_{32} + t_{31}^{(1)} dg_{12}^{(1)} - t_{33} dg_{23}^{(1)} - t_{31}^{(2)} dg_{12}^{(2)} - t_{32}^{(2)} dg_{22}^{(2)}, \\ dv_{33}^{(1)} &= dt_{33} + t_{31}^{(1)} dg_{13}^{(1)} + t_{32}^{(1)} dg_{23}^{(1)} - t_{31}^{(2)} dg_{13}^{(2)} - t_{32}^{(2)} dg_{23}^{(2)} \end{aligned}$$

and

$$\begin{aligned}
dv_{11}^{(2)} &= t_{11}dg_{11}^{(2)}, \quad dv_{12}^{(2)} = t_{11}dg_{12}^{(2)}, \quad dv_{13}^{(2)} = t_{11}dg_{13}^{(2)}, \\
dv_{21}^{(2)} &= dt_{21}^{(2)} + t_{21}^{(1)}dg_{11}^{(2)} + t_{22}dg_{12}^{(2)} - t_{21}^{(2)}dg_{12}^{(1)}, \\
dv_{22}^{(2)} &= t_{21}^{(2)}dg_{12}^{(1)} + t_{21}^{(1)}dg_{12}^{(2)} + t_{22}dg_{22}^{(2)}, \\
dv_{23}^{(2)} &= t_{21}^{(2)}dg_{13}^{(1)} + t_{21}^{(1)}dg_{13}^{(2)} + t_{22}dg_{23}^{(2)}, \\
dv_{31}^{(2)} &= dt_{31}^{(2)} - t_{32}^{(2)}dg_{12}^{(1)} + t_{31}^{(1)}dg_{11}^{(2)} + t_{32}^{(1)}dg_{12}^{(2)} + t_{33}dg_{13}^{(2)}, \\
dv_{32}^{(2)} &= dt_{32}^{(2)} + t_{31}^{(2)}dg_{12}^{(1)} + t_{31}^{(1)}dg_{12}^{(2)} + t_{32}^{(1)}dg_{22}^{(2)} + t_{33}dg_{23}^{(2)}, \\
dv_{33}^{(2)} &= t_{31}^{(2)}dg_{13}^{(1)} + t_{32}^{(2)}dg_{23}^{(1)} + t_{31}^{(1)}dg_{13}^{(2)} + t_{32}^{(1)}dg_{23}^{(2)} + t_{33}dg_{33}^{(2)}.
\end{aligned}$$

According to the definition, we now have  $d\tilde{v}_{jk} = dv_{jk}^{(1)} \wedge dv_{jk}^{(2)}$ , then

$$\begin{aligned}
[d\tilde{V}] &= d\tilde{v}_{11} \wedge d\tilde{v}_{12} \wedge d\tilde{v}_{13} \wedge d\tilde{v}_{21} \wedge d\tilde{v}_{22} \wedge d\tilde{v}_{23} \wedge d\tilde{v}_{31} \wedge d\tilde{v}_{32} \wedge d\tilde{v}_{33} \\
&= \left( \prod_{j=1}^3 t_{jj}^{2(3-j)+1} \right) [d\tilde{T}][d\tilde{G}].
\end{aligned}$$

(ii). Let the diagonal elements of  $\tilde{U}$  be real and all other elements in  $\tilde{U}$  and  $\tilde{T}$  complex. Starting from Eq. (3.31), observing that  $d\tilde{G}$  is  $d\tilde{G}_1$  in this case with the wedge product  $[d\tilde{G}_1]$ , and taking the variables  $d\tilde{v}_{jk}$ 's in the order  $dv_{jk}^{(1)}, j \geq k, dv_{jk}^{(2)}, j < k, dv_{jk}^{(2)}, j \geq k, dv_{jk}^{(1)}, j < k$  and the other variables in the order  $dt_{jk}^{(1)}, j \geq k, dg_{jk}^{(1)}, j > k, dt_{jk}^{(2)}, j \geq k, dg_{jk}^{(2)}, j > k$ , we have the following configuration in the Jacobian matrix:

$$\begin{bmatrix}
\mathbb{1} & * & * & * \\
0 & -A_1 & 0 & A_2 \\
0 & * & \mathbb{1} & * \\
0 & -A_2 & 0 & -A_1
\end{bmatrix}$$

where the matrices marked by  $*$  can be made null by operating with the first and third column submatrices when taking the determinant. Thus they can be taken as null matrices, and  $A_1$  and  $A_2$  are triangular matrices with respectively  $t_{jj}^{(1)}$  and  $t_{jj}^{(2)}$  repeated  $n-j$  times in the diagonal. The Jacobian matrix can be reduced to the form

$$\begin{bmatrix} A & B \\ -B & A \end{bmatrix}, A = \begin{bmatrix} \mathbb{1} & 0 \\ 0 & -A_1 \end{bmatrix}, B = \begin{bmatrix} 0 & 0 \\ 0 & A_2 \end{bmatrix}.$$

Then the determinant is given by

$$\begin{aligned}
\begin{vmatrix} A & B \\ -B & A \end{vmatrix} &= \left| \det((A + \sqrt{-1}B)(A + \sqrt{-1}B)^*) \right| \\
&= \left| \det((-A_1 + \sqrt{-1}A_2)(-A_1 + \sqrt{-1}A_2)^*) \right| \\
&= \prod_{j=1}^p |\tilde{t}_{jj}|^{2(n-j)}
\end{aligned}$$

since  $-A_1 + \sqrt{-1}A_2$  is triangular with the diagonal elements  $-t_{jj}^{(1)} + \sqrt{-1}t_{jj}^{(2)}$  repeated  $n - j$  times, giving  $\left(t_{jj}^{(1)}\right)^2 + \left(t_{jj}^{(2)}\right)^2 = |\tilde{t}_{jj}|^2$  repeated  $n - j$  times in the final determinant and hence the result.  $\square$

By using Proposition 3.16 one can obtain expressions for the integral over  $\tilde{U}$  of  $[d\tilde{G}]$  and  $[d\tilde{G}_1]$ . These will be stated as corollaries here and the proofs will be given after stating both the corollaries.

**Theorem 3.17.** *Let  $d\tilde{G} = d\tilde{U} \cdot \tilde{U}^*$ , where  $\tilde{U} \in \mathcal{U}(n)$ . Then*

$$\text{vol}(\mathcal{U}(n)) = \int_{\mathcal{U}(n)} [d\tilde{G}] = \frac{2^n \pi^{n^2}}{\tilde{\Gamma}_n(n)} = \frac{2^n \pi^{\frac{n(n+1)}{2}}}{1!2! \cdots (n-1)!}. \quad (3.32)$$

*Proof.* Let  $\tilde{X}$  be a  $n \times n$  matrix of independent complex variables. Let

$$B = \int_{\tilde{X}} [d\tilde{X}] e^{-\text{Tr}(\tilde{X}\tilde{X}^*)} = \int_{\tilde{X}} [d\tilde{X}] e^{-\sum_{j,k} |\tilde{x}_{jk}|^2} = \pi^{n^2}$$

since

$$\int_{\tilde{x}_{jk}} e^{-|\tilde{x}_{jk}|^2} d\tilde{x}_{jk} = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{-\left((x_{jk}^{(1)})^2 + (x_{jk}^{(2)})^2\right)} dx_{jk}^{(1)} dx_{jk}^{(2)} = \pi.$$

Consider the transformation used in Proposition 3.16 with  $t_{jj}$ 's real and positive. Then

$$\text{Tr}(\tilde{X}\tilde{X}^*) = \text{Tr}(\tilde{T}\tilde{T}^*)$$

and let

$$B = \int_{\tilde{X}} [d\tilde{X}] e^{-\text{Tr}(\tilde{X}\tilde{X}^*)} = \int_{\tilde{T}} \int_{\tilde{U}} \left( \prod_{j=1}^n t_{jj}^{2(n-j)+1} \right) e^{-\sum_{j \geq k} |\tilde{t}_{jk}|^2} [d\tilde{T}] [d\tilde{G}].$$

But

$$\int_0^\infty t_{jj}^{2(n-j)+1} e^{-t_{jj}^2} dt_{jj} = \frac{1}{2} \Gamma(n-j+1) \quad \text{for } n-j+1 > 0$$

and for  $j > k$

$$\int_{\tilde{t}_{jk}} e^{-|\tilde{t}_{jk}|^2} d\tilde{t}_{jk} = \pi.$$

Then the integral over  $\tilde{T}$  gives

$$2^{-n} \pi^{\frac{n(n-1)}{2}} \prod_{j=1}^n \Gamma(n-j+1) = 2^{-n} \tilde{\Gamma}_n(n).$$

Hence

$$\int_{\mathcal{U}(n)} [\mathrm{d}\tilde{G}] = \frac{2^n \pi^{n^2}}{\tilde{\Gamma}_n(n)}.$$

□

**Theorem 3.18.** Let  $\tilde{U}_1 \in \mathcal{U}(n)$  with the diagonal elements real. Let  $\mathrm{d}\tilde{G}_1 = \mathrm{d}\tilde{U}_1 \cdot \tilde{U}_1^*$ . Let the full unitary group of such  $n \times n$  matrices  $\tilde{U}_1$  be denoted by  $\mathcal{U}_1(n) = \mathcal{U}(n)/\mathcal{U}(1)^{\times n}$ . Then

$$\mathrm{vol}(\mathcal{U}_1(n)) = \mathrm{vol}(\mathcal{U}(n)/\mathcal{U}(1)^{\times n}) = \int_{\mathcal{U}_1(n)} [\mathrm{d}\tilde{G}_1] = \frac{\pi^{n(n-1)}}{\tilde{\Gamma}_n(n)} = \frac{\pi^{\frac{n(n-1)}{2}}}{1!2! \cdots (n-1)!}. \quad (3.33)$$

*Proof.* Now consider the transformation used in Proposition 3.16 with all the elements in  $\tilde{T}$  complex and the diagonal elements of  $\tilde{U}$  real. Then

$$B = \int_{\tilde{T}} \int_{\tilde{U}_1} \left( \prod_{j=1}^n t_{jj}^{2(n-j)} \right) e^{-\sum_{j \geq k} |\tilde{t}_{jk}|^2} [\mathrm{d}\tilde{T}] [\mathrm{d}\tilde{G}_1].$$

Note that

$$\prod_{j>k} \left( \int_{\tilde{t}_{jk}} e^{-|\tilde{t}_{jk}|^2} \mathrm{d}\tilde{t}_{jk} \right) = \pi^{\frac{n(n-1)}{2}}.$$

Let  $\tilde{t}_{jj} = \tilde{t} = t_1 + \sqrt{-1}t_2$ . Put  $t_1 = r \cos \theta$  and  $t_2 = r \sin \theta$ . Let the integral over  $\tilde{t}_{jj}$  be denoted by  $a_j$ . Then

$$\begin{aligned} a_j &= \int_{\tilde{t}} |\tilde{t}|^{2(n-j)} e^{-|\tilde{t}|^2} \mathrm{d}\tilde{t} = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} (t_1^2 + t_2^2)^{n-j} e^{-(t_1^2 + t_2^2)} \mathrm{d}t_1 \mathrm{d}t_2 \\ &= 4 \int_0^{+\infty} \int_0^{+\infty} (t_1^2 + t_2^2)^{n-j} e^{-(t_1^2 + t_2^2)} \mathrm{d}t_1 \mathrm{d}t_2 \\ &= 4 \int_{\theta=0}^{\frac{\pi}{2}} \int_{r=0}^{\infty} (r^2)^{n-j} \cdot e^{-r^2} \cdot r \cdot \mathrm{d}r \mathrm{d}\theta \\ &= \pi \Gamma(n-j+1) \text{ for } n-j+1 > 0. \end{aligned}$$

Then

$$B = \pi^n \tilde{\Gamma}_n(n) \int_{\tilde{U}_1} [\mathrm{d}\tilde{G}_1] = \pi^{n^2}$$

implies that

$$\int_{\tilde{U}_1} [\mathrm{d}\tilde{G}_1] = \frac{\pi^{n(n-1)}}{\tilde{\Gamma}_n(n)},$$

which establishes the result. □

**Example 3.19.** Evaluate the integral

$$\Delta(\alpha) = \int_{\tilde{X}} [\mathrm{d}\tilde{X}] \left| \det(\tilde{X}\tilde{X}^*) \right|^\alpha \cdot e^{-\mathrm{Tr}(\tilde{X}\tilde{X}^*)} = \pi^{n^2} \frac{\tilde{\Gamma}_n(\alpha + n)}{\tilde{\Gamma}_n(n)} \quad (3.34)$$

for  $\mathrm{Re}(\alpha) > -1$ , where  $\tilde{X} \in \mathbb{C}^{n \times n}$  matrix of independent complex variables. Indeed, put  $\tilde{X} = \tilde{T}\tilde{U}$ , where  $\tilde{U} \in \mathcal{U}(n)$ ,  $\tilde{T}$  is lower triangular with real distinct and positive diagonal elements. Then

$$\begin{aligned} \left| \det(\tilde{X}\tilde{X}^*) \right|^\alpha &= \prod_{j=1}^n t_{jj}^{2\alpha}, \\ \mathrm{Tr}(\tilde{X}\tilde{X}^*) &= \mathrm{Tr}(\tilde{T}\tilde{T}^*) = \sum_{j \geq k} |\tilde{t}_{jk}|^2 = \sum_{j=1}^n t_{jj}^2 + \sum_{j > k} |\tilde{t}_{jk}|^2, \\ [\mathrm{d}\tilde{X}] &= \left( \prod_{j=1}^n t_{jj}^{2(n-j)+1} \right) [\mathrm{d}\tilde{T}][\mathrm{d}\tilde{G}], \\ \int_{\tilde{U}} [\mathrm{d}\tilde{G}] &= \frac{2^n \pi^{n^2}}{\tilde{\Gamma}_n(n)}. \end{aligned}$$

Thus

$$\begin{aligned} \Delta(\alpha) &= \int_{\tilde{T}, \tilde{U}} [\mathrm{d}\tilde{T}][\mathrm{d}\tilde{G}] \left( \prod_{j=1}^n t_{jj}^{2\alpha+2(n-j)+1} e^{-t_{jj}^2} \right) e^{-\sum_{j > k} |\tilde{t}_{jk}|^2} \\ &= \left( \prod_{j=1}^n \int_0^\infty t_{jj}^{2\alpha+2(n-j)+1} e^{-t_{jj}^2} \mathrm{d}t_{jj} \right) \cdot \left( \prod_{j > k} \int_{-\infty}^{+\infty} e^{-|\tilde{t}_{jk}|^2} \mathrm{d}\tilde{t}_{jk} \right) \cdot \int_{\tilde{U}} [\mathrm{d}\tilde{G}] \\ &= \left( 2^{-n} \prod_{j=1}^n \Gamma(\alpha + n - j + 1) \right) \cdot \pi^{\frac{n(n-1)}{2}} \cdot \frac{2^n \pi^{n^2}}{\tilde{\Gamma}_n(n)} \end{aligned}$$

for  $\mathrm{Re}(\alpha + n) > n - 1$  or  $\mathrm{Re}(\alpha) > -1$ . That is,

$$\Delta(\alpha) = \pi^{n^2} \frac{\tilde{\Gamma}_n(\alpha + n)}{\tilde{\Gamma}_n(n)} \quad \text{for } \mathrm{Re}(\alpha) > -1.$$

**Proposition 3.20.** Let  $\tilde{X} \in \mathbb{C}^{n \times n}$  be a hermitian matrix of independent complex entries with real distinct eigenvalues  $\lambda_1 > \lambda_2 > \dots > \lambda_n$ . Let  $\tilde{U} \in \mathcal{U}(n)$  with real diagonal entries and let  $\tilde{X} = \tilde{U}D\tilde{U}^*$ , where  $D = \mathrm{diag}(\lambda_1, \dots, \lambda_n)$ . Then

$$[\mathrm{d}\tilde{X}] = \left( \prod_{j > k} |\lambda_k - \lambda_j|^2 \right) \cdot [\mathrm{d}D][\mathrm{d}\tilde{G}_1], \quad (3.35)$$

where  $\mathrm{d}\tilde{G}_1 = \tilde{U}^* \cdot \mathrm{d}\tilde{U}$ .

*Proof.* Take the differentials in  $\tilde{X} = \tilde{U}D\tilde{U}^*$  to get

$$\mathrm{d}\tilde{X} = \mathrm{d}\tilde{U} \cdot D \cdot \tilde{U}^* + \tilde{U} \cdot \mathrm{d}D \cdot \tilde{U}^* + \tilde{U} \cdot D \cdot \mathrm{d}\tilde{U}^*.$$

Premultiply by  $\tilde{U}^*$ , postmultiply by  $\tilde{U}$  and observe that  $d\tilde{G}_1$  is skew hermitian. Then one has

$$d\tilde{W} = d\tilde{G}_1 \cdot D + dD - D \cdot d\tilde{G}_1$$

where  $d\tilde{W} = \tilde{U}^* \cdot d\tilde{X}\tilde{U}$  with  $[d\tilde{W}] = [d\tilde{X}]$ . Using the same steps as in the proof of Proposition 5.10, we have

$$[d\tilde{W}] = \left( \prod_{j>k} |\lambda_k - \lambda_j|^2 \right) [dD][d\tilde{G}_1].$$

Hence the result follows.  $\square$

**Example 3.21.** Let  $D = \text{diag}(\lambda_1, \dots, \lambda_n)$  where the  $\lambda_j$ 's are real distinct and positive or let  $\lambda_1 > \dots > \lambda_n > 0$ . Show that

(i)

$$\int_{\lambda_1 > \dots > \lambda_n > 0} [dD] \left( \prod_{j>k} |\lambda_k - \lambda_j|^2 \right) e^{-\text{Tr}(D)} = \frac{(\tilde{\Gamma}_n(n))^2}{\pi^{n(n-1)}} = \left( \prod_{j=1}^n \Gamma(j) \right)^2;$$

(ii)

$$\int_{\lambda_1 > \dots > \lambda_n > 0} [dD] \left( \prod_{j>k} |\lambda_k - \lambda_j|^2 \right) \cdot \left( \prod_{j=1}^n |\lambda_j|^{\alpha-n} \right) \cdot e^{-\text{Tr}(D)} = \frac{\tilde{\Gamma}_n(\alpha) \tilde{\Gamma}_n(n)}{\pi^{n(n-1)}}.$$

In fact, let  $\tilde{Y}$  be a  $n \times n$  hermitian positive definite matrix,  $\tilde{U}$  a unitary matrix with real diagonal elements such that

$$\tilde{U}^* \tilde{Y} \tilde{U} = D = \text{diag}(\lambda_1, \dots, \lambda_n).$$

From the matrix-variate gamma integral

$$\tilde{\Gamma}_n(\alpha) = \int_{\tilde{Y}=\tilde{Y}^*>0} [d\tilde{Y}] \left| \det(\tilde{Y}) \right|^{\alpha-n} \cdot e^{-\text{Tr}(\tilde{Y})} \quad \text{for } \text{Re}(\alpha) > n-1.$$

Hence

$$\tilde{\Gamma}_n(n) = \int_{\tilde{Y}=\tilde{Y}^*>0} [d\tilde{Y}] e^{-\text{Tr}(\tilde{Y})}.$$

Then  $\tilde{Y} = \tilde{U}D\tilde{U}^*$  implies that

$$[d\tilde{Y}] = \left( \prod_{j>k} |\lambda_k - \lambda_j|^2 \right) [dD][d\tilde{G}_1], \quad d\tilde{G}_1 = \tilde{U}^* \cdot d\tilde{U}.$$

Note that

$$\begin{aligned}\mathrm{Tr}(\tilde{Y}) &= \mathrm{Tr}(\tilde{U}D\tilde{U}^*) = \mathrm{Tr}(D), \\ |\det(\tilde{Y})|^{\alpha-n} &= \left(\prod_{j=1}^n |\lambda_j|\right)^{\alpha-n}.\end{aligned}$$

Then

$$\tilde{\Gamma}_n(n) = \int_{\lambda_1 > \dots > \lambda_n > 0} [dD] \left( \prod_{j > k} |\lambda_k - \lambda_j|^2 \right) e^{-\mathrm{Tr}(D)} \times \int_{\tilde{U}} [d\tilde{G}_1].$$

Clearly

$$\int_{\tilde{U}} [d\tilde{G}_1] = \frac{\pi^{n(n-1)}}{\tilde{\Gamma}_n(n)}.$$

Substituting this, results (i) and (ii) follow.

**Remark 3.22.** We can try to compute the following integral in (3.35):

$$\int_{\tilde{X}:\mathrm{Tr}(\tilde{X})=1} [d\tilde{X}] = \int_{\lambda_1 > \lambda_2 > \dots > \lambda_n > 0} \delta\left(1 - \sum_{j=1}^n \lambda_j\right) \prod_{i < j} (\lambda_i - \lambda_j)^2 \prod_{j=1}^n d\lambda_j \times \int_{\mathcal{U}_1(n)} [d\tilde{G}_1]$$

which is equivalent to the following

$$\begin{aligned}\int_{\tilde{X}:\mathrm{Tr}(\tilde{X})=1} [d\tilde{X}] &= \frac{1}{n!} \int \delta\left(1 - \sum_{j=1}^n \lambda_j\right) \prod_{i < j} (\lambda_i - \lambda_j)^2 \prod_{j=1}^n d\lambda_j \times \int_{\mathcal{U}_1(n)} [d\tilde{G}_1] \\ &= \frac{1}{n!} \frac{\prod_{j=0}^{n-1} \Gamma(n-j)\Gamma(n-j+1)}{\Gamma(n^2)} \times \frac{\pi^{\frac{n(n-1)}{2}}}{\prod_{j=1}^n \Gamma(j)} \\ &= \frac{1}{n!} \pi^{\frac{n(n-1)}{2}} \frac{\Gamma(1)\Gamma(2) \cdots \Gamma(n+1)}{\Gamma(n^2)},\end{aligned}$$

where we used the integral formula:

$$\int \delta\left(1 - \sum_{j=1}^n \lambda_j\right) \prod_{i < j} (\lambda_i - \lambda_j)^2 \prod_{j=1}^n d\lambda_j = \frac{\prod_{j=0}^{n-1} \Gamma(n-j)\Gamma(n-j+1)}{\Gamma(n^2)}.$$

This means that

$$\mathrm{vol}(D(\mathbb{C}^n)) = \pi^{\frac{n(n-1)}{2}} \frac{\Gamma(1)\Gamma(2) \cdots \Gamma(n)}{\Gamma(n^2)}. \quad (3.36)$$

We make some remarks here: to obtain the volume of the set of mixed states acting on  $\mathbb{C}^n$ , one has to integrate the volume element  $[d\tilde{X}]$ . By definition, the first integral gives  $\frac{1}{n!} \frac{1}{C_n^{\text{HS}}}$ , where  $C_n^{\text{HS}} = C_n^{(1,2)}$  and

$$\begin{aligned} \frac{1}{C_n^{(\alpha, \beta)}} &= \overbrace{\int_0^\infty \cdots \int_0^\infty}^n \delta \left( 1 - \sum_{j=1}^n \lambda_j \right) \prod_k^n \lambda_k^{\alpha-1} \prod_{i < j} (\lambda_i - \lambda_j)^\beta \prod_{j=1}^n d\lambda_j \\ &= \frac{1}{\Gamma \left( \alpha n + \beta \frac{n(n-1)}{2} \right)} \prod_{j=1}^n \frac{\Gamma \left( 1 + j \frac{\beta}{2} \right) \Gamma \left( \alpha + (j-1) \frac{\beta}{2} \right)}{\Gamma \left( 1 + \frac{\beta}{2} \right)}, \end{aligned}$$

while the second is equal to the volume of the flag manifold. To make the diagonalization unique, one has to restrict to a certain order of eigenvalues, say  $\lambda_1 > \lambda_2 > \cdots > \lambda_n$  (a generic density matrix is not degenerate), which corresponds to a choice of a certain Weyl chamber of the eigenvalue simplex  $\Delta_{n-1}$ . In other words, different permutations of the vector of  $n$  generically different permutations (Weyl chambers) equals to  $n!$ . This is why the factor  $\frac{1}{n!}$  appears in the right hand side in the above identity. In summary, the transformation

$$\tilde{X} \mapsto (D, \tilde{U})$$

such that  $\tilde{X} = \tilde{U} D \tilde{U}^*$ , is one-to-one if and only if  $D = \text{diag}(\lambda_1, \dots, \lambda_n)$ , where  $\lambda_1 > \cdots > \lambda_n$  and  $\tilde{U} \in \mathcal{U}(n)/\mathcal{U}(1)^{\times n}$ .

**Remark 3.23.** In this remark, we will discuss the connection between two integrals [6]: for  $\alpha, \beta > 0$ ,

$$\begin{aligned} \mathcal{I}_n^{(1)}(\alpha, \beta) &= \overbrace{\int_0^\infty \cdots \int_0^\infty}^n \delta \left( 1 - \sum_{j=1}^n \lambda_j \right) \prod_{k=1}^n \lambda_k^{\alpha-1} \prod_{1 \leq i < j \leq n} (\lambda_i - \lambda_j)^\beta \prod_{j=1}^n d\lambda_j, \\ \mathcal{I}_n^{(w)}(\alpha, \beta) &= \overbrace{\int_0^\infty \cdots \int_0^\infty}^n \exp \left( - \sum_{j=1}^n x_j \right) \prod_{k=1}^n x_k^{\alpha-1} \prod_{1 \leq i < j \leq n} (x_i - x_j)^\beta \prod_{j=1}^n dx_j. \end{aligned}$$

We introduce an auxiliary variable  $t$  in the expression of  $\mathcal{I}_n^{(1)}(\alpha, \beta)$ , and define  $I(t)$  as

$$I(t) = \overbrace{\int_0^\infty \cdots \int_0^\infty}^n \delta \left( t - \sum_{j=1}^n \lambda_j \right) \prod_{k=1}^n \lambda_k^{\alpha-1} \prod_{1 \leq i < j \leq n} (\lambda_i - \lambda_j)^\beta \prod_{j=1}^n d\lambda_j. \quad (3.37)$$

Then  $I(1) = \mathcal{I}_n^{(1)}(\alpha, \beta)$ . Taking the Laplace transform, denoted by  $\mathcal{L}$ , of  $I(t)$ , we obtain

$$\mathcal{L}(I) := \overbrace{\int_0^\infty \cdots \int_0^\infty}^n \left[ \int \delta \left( t - \sum_{j=1}^n \lambda_j \right) e^{-st} dt \right] \prod_{k=1}^n \lambda_k^{\alpha-1} \prod_{1 \leq i < j \leq n} (\lambda_i - \lambda_j)^\beta \prod_{j=1}^n d\lambda_j \quad (3.38)$$

$$= \overbrace{\int_0^\infty \cdots \int_0^\infty}^n \exp \left( -s \sum_{j=1}^n \lambda_j \right) \prod_{k=1}^n \lambda_k^{\alpha-1} \prod_{1 \leq i < j \leq n} (\lambda_i - \lambda_j)^\beta \prod_{j=1}^n d\lambda_j \quad (3.39)$$

$$= s^{-\alpha n - \beta \binom{n}{2}} \overbrace{\int_0^\infty \cdots \int_0^\infty}^n \exp \left( -\sum_{j=1}^n x_j \right) \prod_{k=1}^n x_k^{\alpha-1} \prod_{1 \leq i < j \leq n} (x_i - x_j)^\beta \prod_{j=1}^n dx_j, \quad (3.40)$$

which means that

$$\tilde{I}(s) := \mathcal{L}(I) = s^{-\alpha n - \beta \binom{n}{2}} \cdot \mathcal{I}^{(w)}(\alpha, \beta). \quad (3.41)$$

Therefore

$$I(t) = \mathcal{L}^{-1}(\mathcal{L}(I)) = \mathcal{L}^{-1}(\tilde{I}) = \frac{t^{\alpha n + \beta \binom{n}{2} - 1}}{\Gamma(\alpha n + \beta \binom{n}{2})} \cdot \mathcal{I}^{(w)}(\alpha, \beta). \quad (3.42)$$

Letting  $t = 1$  gives the conclusion:

$$\mathcal{I}_n^{(1)}(\alpha, \beta) = \frac{1}{\Gamma(\alpha n + \beta \binom{n}{2})} \cdot \mathcal{I}^{(w)}(\alpha, \beta). \quad (3.43)$$

In order to calculate the integral  $\mathcal{I}_n^{(1)}(\alpha, \beta)$ , it suffices to calculate the integral  $\mathcal{I}^{(w)}(\alpha, \beta)$ , which is derived in Corollary 6.4 via Selberg's integral (See Appendix):

$$\mathcal{I}^{(w)}(\alpha, \beta) = \prod_{j=1}^n \frac{\Gamma \left( 1 + j \frac{\beta}{2} \right) \Gamma \left( \alpha + (j-1) \frac{\beta}{2} \right)}{\Gamma \left( 1 + \frac{\beta}{2} \right)}. \quad (3.44)$$

Matrix integrals, especially over unitary groups, are very important. We recently present some results of this aspect [19]. Generally speaking, the computation of matrix integrals is very difficult from the first principle. Thus frequently we need to perform variable substitution in computing integrals. The first step in substitution is to compute Jacobians of this transformation, this is what we present. We also apply the matrix integrals over unitary groups to a problem [20, 21] in quantum information theory.

## 4 Applications

### 4.1 Hilbert-Schmidt volume of the set of mixed quantum states

The present subsection is directly written based on [24, 25]. The results were already obtained. We just here add some interpretation, from my angle, about them since the details concerning

computation therein are almost ignored. Any unitary matrix may be considered as an element of the Hilbert-Schmidt space of operators with the scalar product  $\langle \tilde{U}, \tilde{V} \rangle_{\text{HS}} = \text{Tr}(\tilde{U}^* \tilde{V})$ . This suggests the following definition of an invariant metric of the unitary group  $\mathcal{U}(n)$ : denote  $d\tilde{G} := \tilde{U}^* d\tilde{U}$ , then

$$ds^2 := \langle d\tilde{G}, d\tilde{G} \rangle_{\text{HS}} = -\text{Tr}(d\tilde{G}^2), \quad (4.1)$$

implying

$$ds^2 = \sum_{i,j=1}^n |d\tilde{G}_{ij}|^2 = \sum_{j=1}^n |d\tilde{G}_{jj}|^2 + 2 \sum_{i<j}^n |d\tilde{G}_{ij}|^2. \quad (4.2)$$

Since  $d\tilde{G}^* = -d\tilde{G}$ , it follows that

$$ds^2 = \sum_{j=1}^n |d\tilde{G}_{jj}|^2 + 2 \sum_{i<j}^n \left( d(\text{Re}(\tilde{G}_{ij})) \right)^2 + 2 \sum_{i<j}^n \left( d(\text{Im}(\tilde{G}_{ij})) \right)^2. \quad (4.3)$$

This indicates that the Hilbert-Schmidt volume element is given by

$$d\nu = 2^{\frac{n(n-1)}{2}} \prod_{j=1}^n d(\text{Im}(\tilde{G}_{jj})) \times \prod_{i<j} d(\text{Re}(\tilde{G}_{ij})) d(\text{Im}(\tilde{G}_{ij})) = 2^{\frac{n(n-1)}{2}} [d\tilde{G}], \quad (4.4)$$

that is,

$$\text{vol}_{\text{HS}}(\mathcal{U}(n)) := \int_{\mathcal{U}(n)} d\nu = 2^{\frac{n(n-1)}{2}} \int_{\mathcal{U}(n)} [d\tilde{G}] \quad (4.5)$$

$$= 2^{\frac{n(n-1)}{2}} \times \frac{2^n \pi^{\frac{n(n+1)}{2}}}{1!2! \cdots (n-1)!} \quad (4.6)$$

$$= \frac{(2\pi)^{\frac{n(n+1)}{2}}}{1!2! \cdots (n-1)!}. \quad (4.7)$$

Finally we have obtained the Hilbert-Schmidt volume of unitary group:

$$\text{vol}_{\text{HS}}(\mathcal{U}(n)) = 2^{\frac{n(n-1)}{2}} \text{vol}(\mathcal{U}(n)) = \frac{(2\pi)^{\frac{n(n+1)}{2}}}{1!2! \cdots (n-1)!}. \quad (4.8)$$

We compute the volume of the convex  $(n^2 - 1)$ -dimensional set  $D(\mathbb{C}^n)$  of density matrices of size  $n$  with respect to the Hilbert-Schmidt measure.

The set of mixed quantum states  $D(\mathbb{C}^n)$  consists of Hermitian, positive matrices of size  $n$ , normalized by the trace condition

$$D(\mathbb{C}^n) = \{\tilde{\rho} : \mathbb{C}^n \rightarrow \mathbb{C}^n | \tilde{\rho}^* = \tilde{\rho}, \tilde{\rho} \geq 0, \text{Tr}(\tilde{\rho}) = 1\}. \quad (4.9)$$

It is a compact convex set of dimensionality  $(n^2 - 1)$ . Any density matrix may be diagonalized by a unitary rotation,  $\tilde{\rho} = \tilde{U}\Lambda\tilde{U}^*$ , where  $\tilde{U} \in \mathcal{U}(n)$  and  $\Lambda = \text{diag}(\lambda_1, \dots, \lambda_n)$  for  $\lambda_j \in \mathbb{R}^+$ . Since  $\text{Tr}(\tilde{\rho}) = 1$ , it follows that  $\sum_{j=1}^n \lambda_j = 1$ , so the spectra space is isomorphic with a  $(n - 1)$ -dimensional probability simplex  $\Delta_{n-1} := \left\{ p \in \mathbb{R}^+ : \sum_{j=1}^n p_j = 1 \right\}$ .

Let  $\tilde{B}$  be a diagonal unitary matrix. Since  $\tilde{\rho} = \tilde{U}\tilde{B}\Lambda\tilde{B}^*\tilde{U}^*$ , in the generic case of a *non-degenerate spectrum* (i.e. with distinct non-negative eigenvalues), the unitary matrix  $\tilde{U}$  is determined up to  $n$  arbitrary phases entering  $\tilde{B}$ . On the other hand, the matrix  $\Lambda$  is defined up to a permutation of its entries. The form of the set of all such permutations depends on the character of the degeneracy of the spectrum of  $\tilde{\rho}$ .

Representation  $\tilde{\rho} = \tilde{U}\tilde{B}\Lambda\tilde{B}^*\tilde{U}^*$  makes the description of some topological properties of the  $(n^2 - 1)$ -dimensional space  $D(\mathbb{C}^n)$  easier. Identifying points in  $\Delta_{n-1}$  which have the same components (but ordered in a different way), we obtain an asymmetric simplex  $\tilde{\Delta}_{n-1}$ . Equivalently, one can divide  $\Delta_{n-1}$  into  $n!$  identical simplexes and take any one of them. The asymmetric simplex  $\tilde{\Delta}_{n-1}$  can be decomposed in the following natural way:

$$\tilde{\Delta}_{n-1} = \bigcup_{d_1 + \dots + d_k = n} \delta_{d_1, \dots, d_k}, \quad (4.10)$$

where  $k = 1, \dots, n$  denotes the number of different coordinates of a given point of  $\tilde{\Delta}_{n-1}$ ,  $d_1$  the number of occurrences of the largest coordinate,  $d_2$  the number of occurrences of the second largest etc. Observe that  $\delta_{d_1, \dots, d_k}$  is homeomorphic with the set  $G_k$ , where  $G_1$  is a single point,  $G_2$  is a half-closed interval,  $G_3$  an open triangle with one edge but without corners and, generally,  $G_k$  is an  $(k - 1)$ -dimensional simplex with one  $(k - 2)$ -dimensional hyperface without boundary (the latter is homeomorphic with an  $(k - 2)$ -dimensional open simplex). There are  $n$  ordered eigenvalues:  $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$ , and  $n - 1$  independent relation operators "larger(>) or equal(=)", which makes altogether  $2^{n-1}$  different possibilities. Thus,  $\tilde{\Delta}_{n-1}$  consists of  $2^{n-1}$  parts, out of which  $\binom{n-1}{m-1}$  parts are homeomorphic with  $G_m$ , when  $m$  ranges from 1 to  $n$ .

Let us denote the part of the space  $D(\mathbb{C}^n)$  related to the spectrum in  $\delta_{d_1, \dots, d_k}$  ( $k$  different eigenvalues; the largest eigenvalue has  $d_1$  multiplicity, the second largest  $d_2$  etc) by  $D_{d_1, \dots, d_k}$ . A mixed state  $\tilde{\rho}$  with this kind of the spectrum remains invariant under arbitrary unitary rotations performed in each of the  $d_j$ -dimensional subspaces of degeneracy. Therefore the unitary matrix  $\tilde{B}$  has a block diagonal structure with  $k$  blocks of size equal to  $d_1, \dots, d_k$  and

$$D_{d_1, \dots, d_k} \sim [\mathcal{U}(n) / (\mathcal{U}(d_1) \times \dots \times \mathcal{U}(d_k))] \times G_k, \quad (4.11)$$

where  $d_1 + \dots + d_k = n$  and  $d_j > 0$  for  $j = 1, \dots, k$ . Thus  $D(\mathbb{C}^n)$  has the structure

$$D(\mathbb{C}^n) \sim \bigcup_{d_1 + \dots + d_k = n} D_{d_1, \dots, d_k} \sim \bigcup_{d_1 + \dots + d_k = n} [\mathcal{U}(n) / (\mathcal{U}(d_1) \times \dots \times \mathcal{U}(d_k))] \times G_k, \quad (4.12)$$

where the sum ranges over all partitions  $(d_1, \dots, d_k) \vdash n$  of  $n$ . The group of rotation matrices  $\tilde{B}$  equivalent to  $\mathcal{U}(d_1) \times \dots \times \mathcal{U}(d_k)$  is called the *stability group* of  $\mathcal{U}(n)$ .

Note also that the part of  $D_{1,\dots,1}$  represents a generic, non-degenerate spectrum. In this case all elements of the spectrum of  $\tilde{\rho}$  are different and the stability group is equivalent to an  $n$ -torus

$$D_{1,\dots,1} \sim [\mathcal{U}(n)/(\mathcal{U}(1)^{\times n})] \times G_n. \quad (4.13)$$

The above representation of generic states enables us to define a *product measure* in the space  $D(\mathbb{C}^n)$  of mixed quantum states. To this end, one can take the uniform (Haar) measure on  $\mathcal{U}(n)$  and a certain measure on the simplex  $\Delta_{n-1}$ .

The other  $2^{n-1} - 1$  parts of  $D(\mathbb{C}^n)$  represent various kinds of degeneracy and have *measure zero*. The number of non-homeomorphic parts is equal to the number  $P(n)$  of different representations of the number  $n$  as the sum of positive natural numbers. Thus  $P(n)$  gives the number of different topological structures present in the space  $D(\mathbb{C}^n)$ .

To specify uniquely the unitary matrix of eigenvectors  $\tilde{U}$ , it is thus sufficient to select a point on the coset space

$$\text{Fl}_{\mathbb{C}}^{(n)} := \mathcal{U}(n)/\mathcal{U}(1)^{\times n},$$

called the *complex flag manifold*. The volume of this complex flag manifold is:

$$\text{vol}_{\text{HS}} \left( \text{Fl}_{\mathbb{C}}^{(n)} \right) = \frac{\text{vol}_{\text{HS}} (\mathcal{U}(n))}{\text{vol}_{\text{HS}} (\mathcal{U}(1))^n} = \frac{(2\pi)^{\frac{n(n-1)}{2}}}{1!2! \dots (n-1)!}. \quad (4.14)$$

The generic density matrix is thus determined by  $(n-1)$  parameters determining eigenvalues and  $(n^2 - n)$  parameters related to eigenvectors, which sum up to the dimensionality  $(n^2 - 1)$  of  $D(\mathbb{C}^n)$ . Although for degenerate spectra the dimension of the flag manifold decreases, these cases of measure zero do not influence the estimation of the volume of the entire set of density matrices. In this subsection, we shall use the Hilbert-Schmidt metric. The infinitesimal distance takes a particularly simple form

$$ds_{\text{HS}}^2 = \|\text{d}\tilde{\rho}\|_{\text{HS}}^2 = \langle \text{d}\tilde{\rho}, \text{d}\tilde{\rho} \rangle_{\text{HS}} \quad (4.15)$$

valid for any dimension  $n$ . Making use of the diagonal form  $\tilde{\rho} = \tilde{U}\Lambda\tilde{U}^\dagger$ , we may write

$$\text{d}\tilde{\rho} = \tilde{U} \left( \text{d}\Lambda + [\tilde{U}\text{d}\tilde{U}, \Lambda] \right) \tilde{U}^* \quad (4.16)$$

Thus the infinitesimal distance can be rewritten as

$$ds_{\text{HS}}^2 = \sum_{j=1}^n d\lambda_j^2 + 2 \sum_{i<j}^n (\lambda_i - \lambda_j)^2 \left| \langle i | \tilde{U}^* d\tilde{U} | j \rangle \right|^2 \quad (4.17)$$

$$= \sum_{j=1}^n d\lambda_j^2 + 2 \sum_{i<j}^n (\lambda_i - \lambda_j)^2 \left| \langle i | d\tilde{G} | j \rangle \right|^2, \quad (4.18)$$

where  $d\tilde{G} = \tilde{U}^* d\tilde{U}$ . Apparently,  $\sum_{j=1}^n d\lambda_j = 0$  since  $\sum_{j=1}^n \lambda_j = 1$ . Thus

$$ds_{\text{HS}}^2 = \sum_{i,j=1}^{n-1} d\lambda_i m_{ij} d\lambda_j + 2 \sum_{i<j}^n (\lambda_i - \lambda_j)^2 \left| d\tilde{G}_{ij} \right|^2. \quad (4.19)$$

The corresponding volume element gains a factor  $\sqrt{\det(M)}$ , where  $M = [m_{ij}]$  is the metric in the  $(n^2 - n)$ -dimensional simplex  $\Delta_{n-1}$  of eigenvalues. Note that

$$M = \mathbb{1}_n + \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{bmatrix}.$$

Therefore the Hilbert-Schmidt volume element is given by

$$dV_{\text{HS}} = \sqrt{n} \prod_{j=1}^{n-1} d\lambda_j \prod_{i<j} (\lambda_i - \lambda_j)^2 \left| \prod_{i<j} 2d(\operatorname{Re}(\tilde{G}_{ij})) d(\operatorname{Im}(\tilde{G}_{ij})) \right|. \quad (4.20)$$

Then

$$\begin{aligned} \int dV_{\text{HS}} &= \sqrt{n} 2^{\frac{n(n-1)}{2}} \int \prod_{i<j} (\lambda_i - \lambda_j)^2 \prod_{j=1}^{n-1} d\lambda_j \times \int [d\tilde{G}_1] \\ &= \sqrt{n} 2^{\frac{n(n-1)}{2}} \operatorname{vol}(\mathcal{D}(\mathbb{C}^n)). \end{aligned}$$

That is, respect to Hilbert-Schmidt measure, the volume of the set of mixed quantum states is

$$\operatorname{vol}_{\text{HS}}(\mathcal{D}(\mathbb{C}^n)) = \sqrt{n} 2^{\frac{n(n-1)}{2}} \operatorname{vol}(\mathcal{D}(\mathbb{C}^n)),$$

i.e.

$$\operatorname{vol}_{\text{HS}}(\mathcal{D}(\mathbb{C}^n)) = \sqrt{n} (2\pi)^{\frac{n(n-1)}{2}} \frac{\Gamma(1)\Gamma(2)\cdots\Gamma(n)}{\Gamma(n^2)}. \quad (4.21)$$

We see from the above discussion that the obtained formula of volume depends the used measure. If we used the Hilbert-Schmidt measure, then we get the Hilbert-Schmidt volume of the set of quantum states [25]; if we used the Bures measure, then we get the Bures volume of the set of quantum states [16].

A special important problem is to compute the volume of the set of all separable quantum states, along this line, some investigation on this topic had already been made [22, 23]. There are some interesting topics for computing volumes of the set of some kinds of states, for instance, Milz also considered the volumes of conditioned bipartite state spaces [13], Link gave the geometry of Gaussian quantum states [7] as well. We can also propose some problems like this. Consider the following set of all states being of the form:

$$\mathcal{C} \left( \frac{1}{2}\mathbb{1}_2, \frac{1}{2}\mathbb{1}_2 \right) := \left\{ \tilde{\rho}_{12} \in D(\mathbb{C}^2 \otimes \mathbb{C}^2) : \text{Tr}_1(\tilde{\rho}_{12}) = \frac{1}{2}\mathbb{1}_2 = \text{Tr}_2(\tilde{\rho}_{12}) \right\}.$$

Paratharathy characterized the extremal points of this set [14]. He obtained that all the extremal points of this convex set is maximal entangled states. That is,

$$\frac{|0\rangle|\psi_0\rangle + |1\rangle|\psi_1\rangle}{\sqrt{2}}.$$

It remains open to compute the volume of this convex set  $\mathcal{C} \left( \frac{1}{2}\mathbb{1}_2, \frac{1}{2}\mathbb{1}_2 \right)$ .

## 4.2 Area of the boundary of the set of mixed states

The boundary of the set of mixed states is far from being trivial. Formally it may be written as a solution of the equation

$$\det(\tilde{\rho}) = 0$$

which contains all matrices of a lower rank. The boundary  $\partial D(\mathbb{C}^n)$  contains orbits of different dimensionality generated by spectra of different rank and degeneracy. Fortunately all of them are of measure *zero* besides the generic orbits created by unitary rotations of diagonal matrices with all eigenvalues different and one of them equal to zero;

$$\Lambda = \{0, \lambda_2 < \dots < \lambda_n\}.$$

Such spectra form the  $(n-2)$ -dimensional simplex  $\Delta_{n-2}$ , which contains  $(n-1)!$  the Weyl chambers—this is the number of possible permutations of elements of  $\Lambda$  which all belong to the same unitary orbits.

Hence the hyper-area of the boundary may be computed in a way analogous to (4.21):

$$\begin{aligned}
\int_{\text{rank}(\tilde{X})=n-1} [d\tilde{X}] &= \int_{0 < \lambda_2 < \dots < \lambda_n} \delta \left( \sum_{j=2}^n \lambda_j - 1 \right) \prod_{2=i < j \leq n} |\lambda_i - \lambda_j|^2 \prod_{j=2}^n (\lambda_j^2 d\lambda_j) \times \int_{\mathcal{U}_1(n)} [d\tilde{G}_1] \\
&= \frac{1}{(n-1)!} \overbrace{\int_0^\infty \dots \int_0^\infty}^{n-1} \delta \left( \sum_{j=2}^n \lambda_j - 1 \right) \prod_{2=i < j \leq n} |\lambda_i - \lambda_j|^2 \prod_{j=2}^n (\lambda_j^2 d\lambda_j) \times \int_{\mathcal{U}_1(n)} [d\tilde{G}_1] \\
&= \frac{1}{\Gamma(n)} \frac{\Gamma(1) \dots \Gamma(n) \Gamma(1) \dots \Gamma(n+1)}{\Gamma(n^2-1)} \frac{\pi^{\frac{n(n-1)}{2}}}{\Gamma(1) \dots \Gamma(n)} \\
&= \pi^{\frac{n(n-1)}{2}} \frac{\Gamma(1) \dots \Gamma(n+1)}{\Gamma(n) \Gamma(n^2-1)} = \text{vol}^{(n-1)} \left( \mathcal{D} \left( \mathbb{C}^d \right) \right),
\end{aligned}$$

i.e.

$$\text{vol}_{\text{HS}}^{(n-1)} = \sqrt{n-1} 2^{\frac{n(n-1)}{2}} \text{vol}^{(n-1)} \left( \mathcal{D} \left( \mathbb{C}^d \right) \right) = \sqrt{n-1} (2\pi)^{\frac{n(n-1)}{2}} \frac{\Gamma(1) \dots \Gamma(n+1)}{\Gamma(n) \Gamma(n^2-1)}.$$

In an analogous way, we may find the volume of edges, formed by the unitary orbits of the vector of eigenvalues with two zeros. More generally, states of rank  $N - n$  are unitarily similar to diagonal matrices with  $n$  eigenvalues vanishing,

$$\Lambda = \{ \lambda_1 = \dots = \lambda_m = 0, \lambda_{m+1} < \dots < \lambda_n \}.$$

These edges of order  $m$  are  $n^2 - m^2 - 1$  dimensional, since the dimension of the set of such spectra is  $n - m - 1$ , while the orbits have the structure of  $\mathcal{U}(n) / (\mathcal{U}(m) \times \mathcal{U}(1)^{n-m})$  and dimensionality  $n^2 - m^2 - (n - m)$ . We obtain the volume of the hyperedges

$$\text{vol}_{\text{HS}}^{(n-m)} = \frac{\sqrt{n-m}}{(n-m)!} \frac{1}{C_{n-m}^{(1+2m,2)}} \frac{\text{vol}_{\text{HS}} \left( \text{Fl}_{\mathbb{C}}^{(n)} \right)}{\text{vol}_{\text{HS}} \left( \text{Fl}_{\mathbb{C}}^{(m)} \right)}$$

### 4.3 Volume of a metric ball in unitary group

Consider a metric ball around the identity  $\mathbb{1}_n$  in the  $n$ -dimensional unitary group  $\mathcal{U}(n)$  with Euclidean distance  $\epsilon$ ,

$$B_\epsilon := \left\{ \tilde{U} \in \mathcal{U}(n) : \left\| \tilde{U} - \mathbb{1}_n \right\|_2 \leq \epsilon \right\}, \quad (4.22)$$

where  $\|\cdot\|_p$  is the  $p$ -norm for  $p = 2$ . We consider the invariant Haar-measure  $\mu$ , a uniform distribution defined over  $\mathcal{U}(n)$ . Denote the eigenvalues of  $\tilde{U}$  by  $e^{\sqrt{-1}\theta_j}$ . The joint density of the angles  $\theta_j$  is given by

$$p(\theta_1, \dots, \theta_n) = \frac{1}{(2\pi)^n n!} \prod_{1 \leq i < j \leq n} \left| e^{\sqrt{-1}\theta_i} - e^{\sqrt{-1}\theta_j} \right|^2, \quad (4.23)$$

where  $\theta_j \in [-\pi, \pi], j = 1, \dots, n$ . In what follows, we check the correctness of the integral formula:

$$\int p(\theta) d\theta = 1.$$

Indeed, set  $J(\theta) = \prod_{1 \leq i < j \leq n} \left| e^{\sqrt{-1}\theta_i} - e^{\sqrt{-1}\theta_j} \right|^2$  and  $\zeta_j = e^{\sqrt{-1}\theta_j}$ , so

$$J(\theta) = \prod_{i < j} |\zeta_i - \zeta_j|^2 = \prod_{i < j} (\zeta_i - \zeta_j)(\zeta_i^{-1} - \zeta_j^{-1}) \quad (4.24)$$

$$= (\text{sign } \tau)(\zeta_1 \cdots \zeta_n)^{-(n-1)} \prod_{i < j} (\zeta_i - \zeta_j)^2, \quad (4.25)$$

where  $\tau = (n \cdots 21)$ , i.e.  $\tau(j) = n + 1 - j$  or  $\tau$  is written as

$$\tau := \begin{pmatrix} 1 & 2 & \cdots & n \\ n & n-1 & \cdots & 1 \end{pmatrix}.$$

Note that  $\text{sign } \tau = (-1)^{\frac{n(n-1)}{2}}$ . We see that the integral is the constant term in

$$C_n(\text{sign } \tau)(\zeta_1 \cdots \zeta_n)^{-(n-1)} \prod_{i < j} (\zeta_i - \zeta_j)^2. \quad (4.26)$$

Thus our task is to identify the constant term in this Laurent polynomial. To work on the last factor, we recognize

$$V(\zeta) = \prod_{i < j} (\zeta_i - \zeta_j) \quad (4.27)$$

as a Vandermonde determinant; hence

$$V(\zeta) = \sum_{\sigma \in S_n} (\text{sign } \sigma) \zeta_1^{\sigma(1)-1} \cdots \zeta_n^{\sigma(n)-1}. \quad (4.28)$$

Hence

$$\prod_{i < j} (\zeta_i - \zeta_j)^2 = V(\zeta)^2 = \sum_{\sigma, \pi \in S_n} (\text{sign } \sigma)(\text{sign } \pi) \zeta_1^{\sigma(1)+\pi(1)-2} \cdots \zeta_n^{\sigma(n)+\pi(n)-2}. \quad (4.29)$$

Let us first identify the constant term in

$$J(\theta) = (\text{sign } \tau)(\zeta_1 \cdots \zeta_n)^{-(n-1)} V(\zeta)^2. \quad (4.30)$$

We see this constant term is equal to

$$\begin{aligned}
& \frac{1}{(2\pi)^n} \overbrace{\int_0^{2\pi} \cdots \int_0^{2\pi}}^n J(\theta) d\theta \\
&= (\text{sign } \tau) \frac{1}{(2\pi)^n} \overbrace{\int_0^{2\pi} \cdots \int_0^{2\pi}}^n \left( \sum_{\sigma, \pi \in S_n} (\text{sign } \sigma)(\text{sign } \pi) \zeta_1^{\sigma(1)+\pi(1)-n-1} \cdots \zeta_n^{\sigma(n)+\pi(n)-n-1} \right) d\theta \\
&= (\text{sign } \tau) \sum_{\sigma, \pi \in S_n} (\text{sign } \sigma)(\text{sign } \pi) \left( \frac{1}{2\pi} \int_0^{2\pi} \zeta_1^{\sigma(1)+\pi(1)-n-1} d\theta_1 \right) \times \cdots \times \left( \frac{1}{2\pi} \int_0^{2\pi} \zeta_n^{\sigma(n)+\pi(n)-n-1} d\theta_n \right) \\
&= (\text{sign } \tau) \sum_{(\sigma, \pi) \in S_n \times S_n: \forall j, \sigma(j) + \pi(j) = n+1} (\text{sign } \sigma)(\text{sign } \pi) = (\text{sign } \tau) \sum_{(\sigma, \pi) \in S_n \times S_n: \pi = \tau\sigma} (\text{sign } \sigma)(\text{sign } \pi).
\end{aligned}$$

Note that the sum is over all  $(\sigma, \pi) \in S_n \times S_n$  such that  $\sigma(j) + \pi(j) = n+1$  for each  $j \in \{1, \dots, n\}$ . In other words, we get  $\pi(j) = n+1 - \sigma(j) = \tau(\sigma(j))$  for all  $j \in \{1, \dots, n\}$ , i.e.  $\pi = \tau\sigma$ . Thus the sum is equal to

$$(\text{sign } \tau) \sum_{\sigma \in S_n} (\text{sign } \sigma)(\text{sign } \tau\sigma) = n!,$$

which gives rise to

$$\frac{1}{(2\pi)^n} \overbrace{\int_0^{2\pi} \cdots \int_0^{2\pi}}^n J(\theta) d\theta = \frac{1}{n!}.$$

Now the condition on the distance measure  $\|\tilde{U} - \mathbb{1}_n\|_2 \leq \epsilon$  is equivalent to

$$\sum_{j=1}^n \left| e^{\sqrt{-1}\theta_j} - 1 \right|^2 \leq \epsilon^2.$$

Using Euler's formula  $e^{\sqrt{-1}\theta} = \cos \theta + \sqrt{-1} \sin \theta$  and the fact that  $(\cos \theta - 1)^2 + \sin^2 \theta = 4 \sin^2 \left( \frac{\theta}{2} \right)$ , we get

$$\|\tilde{U} - \mathbb{1}_n\|_2 \leq \epsilon \iff \sum_{j=1}^n \sin^2 \left( \frac{\theta_j}{2} \right) \leq \frac{\epsilon^2}{4}. \quad (4.31)$$

Thus the (normalized) volume of the metric ball  $B_\epsilon$  equals the following:

$$\text{vol}(B_\epsilon) := \mu(B_\epsilon) = \int_{B_\epsilon} d\mu(\tilde{U}). \quad (4.32)$$

By spectral decomposition of unitary matrix, we have

$$\tilde{U} = \tilde{V} \tilde{D} \tilde{V}^*, \quad \tilde{D} = e^{\sqrt{-1}\Theta}, \quad \Theta = \begin{bmatrix} \theta_1 & 0 & \cdots & 0 \\ 0 & \theta_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & \theta_n \end{bmatrix}, \quad \theta_j \in [-\pi, \pi] (j = 1, \dots, n).$$

Hence

$$\tilde{V}^* \cdot d\tilde{U} \cdot \tilde{V} = d\tilde{D} + [\tilde{V}^* d\tilde{V}, \tilde{D}],$$

implying that

$$\tilde{V}^* \cdot \tilde{U}^* d\tilde{U} \cdot \tilde{V} = \tilde{D}^* \left( d\tilde{D} + [\tilde{V}^* d\tilde{V}, \tilde{D}] \right).$$

Let  $d\tilde{G} = \tilde{U}^* d\tilde{U}$ ,  $d\tilde{G}_1 = \tilde{V}^* d\tilde{V}$  and  $d\tilde{X} = \tilde{V}^* \cdot d\tilde{G} \cdot \tilde{V}$ . Thus

$$[d\tilde{X}] = [d\tilde{G}] \quad (4.33)$$

because of  $\tilde{V} \in \mathcal{U}(n)/\mathcal{U}(1)^{\times n}$ . We also have

$$d\tilde{X} = \tilde{D}^* \cdot \left( d\tilde{D} + [d\tilde{G}_1, \tilde{D}] \right) = \tilde{D}^* \cdot \tilde{V}^* \cdot d\tilde{U} \cdot \tilde{V},$$

it follows that

$$[d\tilde{X}] = [d\tilde{U}]. \quad (4.34)$$

Apparently,

$$B_\epsilon = \left\{ \tilde{V} \tilde{D} \tilde{V}^* \in \mathcal{U}(n) : \tilde{D} \in \mathcal{U}(n), \tilde{V} \in \mathcal{U}(n)/\mathcal{U}(1)^{\times n}, \left\| \tilde{D} - \mathbf{1}_n \right\|_2 \leq \epsilon \right\}. \quad (4.35)$$

But

$$[d\tilde{U}] = \prod_{1 \leq i < j \leq n} \left| e^{\sqrt{-1}\theta_i} - e^{\sqrt{-1}\theta_j} \right|^2 [d\tilde{D}] [d\tilde{G}_1], \quad (4.36)$$

therefore

$$[d\tilde{G}] = \prod_{1 \leq i < j \leq n} \left| e^{\sqrt{-1}\theta_i} - e^{\sqrt{-1}\theta_j} \right|^2 [d\tilde{D}] [d\tilde{G}_1], \quad (4.37)$$

together with the facts that the region in which  $(\theta_1, \dots, \theta_n)$  lies is symmetric and  $\tilde{V} \in \mathcal{U}(n)/\mathcal{U}(1)^{\times n}$ , implying

$$\int_{B_\epsilon} [d\tilde{G}] = \frac{1}{n!} \int \cdots \int_{\substack{\theta_j \in [-\pi, \pi] (j=1, \dots, n); \\ \sum_{j=1}^n \sin^2(\theta_j/2) \leq \frac{\epsilon^2}{4}}} \prod_{1 \leq i < j \leq n} \left| e^{\sqrt{-1}\theta_i} - e^{\sqrt{-1}\theta_j} \right|^2 [d\tilde{D}] \times \int_{\mathcal{U}(n)} [d\tilde{G}_1].$$

That is,

$$\int_{B_\epsilon} d\mu(\tilde{U}) = \frac{1}{(2\pi)^n n!} \int \cdots \int_{\substack{\theta_j \in [-\pi, \pi] (j=1, \dots, n); \\ \sum_{j=1}^n \sin^2(\theta_j/2) \leq \frac{\epsilon^2}{4}}} \prod_{1 \leq i < j \leq n} \left| e^{\sqrt{-1}\theta_i} - e^{\sqrt{-1}\theta_j} \right|^2 [d\tilde{D}],$$

where

$$d\mu(\tilde{U}) = \frac{[d\tilde{G}]}{\int_{\mathcal{U}(n)} [d\tilde{G}]} \text{ and } \int_{\mathcal{U}(n)} [d\tilde{G}] = (2\pi)^n \int_{\mathcal{U}(n)} [d\tilde{G}_1].$$

From this, we get

$$\text{vol}(B_\epsilon) = \int \cdots \int_{\substack{\theta_j \in [-\pi, \pi] (j=1, \dots, n); \\ \sum_{j=1}^n \sin^2(\theta_j/2) \leq \frac{\epsilon^2}{4}}} p(\theta_1, \dots, \theta_n) \prod_{j=1}^n d\theta_j, \quad (4.38)$$

where  $\epsilon \in [0, 2\sqrt{n}]$ . For the maximal distance  $\epsilon = 2\sqrt{n}$ , the restriction  $\sum_{j=1}^n \sin^2(\theta_j/2) \leq \frac{\epsilon^2}{4}$  becomes irrelevant and  $\text{vol}(B_{2\sqrt{n}}) = 1$ .

We start by rewriting the  $n$ -dimensional integral (4.38), with the help of a Dirac delta function, as

$$\text{vol}(B_\epsilon) = \int_0^{\frac{\epsilon^2}{4}} \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} \delta\left(t - \sum_{j=1}^n \sin^2\left(\frac{\theta_j}{2}\right)\right) p(\theta_1, \dots, \theta_n) \prod_{j=1}^n d\theta_j dt. \quad (4.39)$$

We know that

$$\int_{-\infty}^{\infty} \delta(t-a) f(t) dt = f(a).$$

If  $f(t) \equiv \mathbb{1}_{\{t:\alpha \leq t \leq \beta\}}$ , then

$$\int_{\alpha}^{\beta} \delta(t-a) dt = \int_{-\infty}^{\infty} \delta(t-a) \mathbb{1}_{\{t:\alpha \leq t \leq \beta\}} dt = \mathbb{1}_{\{t:\alpha \leq t \leq \beta\}}.$$

By using the Fourier representation of Dirac Delta function

$$\delta(t-a) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{\sqrt{-1}(t-a)s} ds,$$

we get

$$\begin{aligned} \int_0^{\frac{\epsilon^2}{4}} \delta(t-a) dt &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left( \int_0^{\frac{\epsilon^2}{4}} e^{\sqrt{-1}(t-a)s} dt \right) ds \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\sqrt{-1} \left( 1 - e^{\sqrt{-1}\frac{\epsilon^2}{4}s} \right)}{se^{\sqrt{-1}as}} ds. \end{aligned}$$

Let  $a = \sum_{j=1}^n \sin^2\left(\frac{\theta_j}{2}\right) = \frac{n}{2} - \sum_{j=1}^n \frac{1}{2} \cos \theta_j$ . Indeed,

$$\begin{aligned} \frac{n}{2} - \sum_{j=1}^n \sin^2(\theta_j/2) &= -\frac{n}{2} + \sum_{j=1}^n (1 - \sin^2(\theta_j/2)) = -\frac{n}{2} + \sum_{j=1}^n \cos^2(\theta_j/2) \\ &= -\frac{n}{2} + \sum_{j=1}^n \frac{1}{2} (1 + \cos \theta_j) = \sum_{j=1}^n \frac{1}{2} \cos \theta_j. \end{aligned}$$

Therefore

$$\begin{aligned} \int_0^{\frac{\epsilon^2}{4}} \delta\left(t - \sum_{j=1}^n \sin^2\left(\frac{\theta_j}{2}\right)\right) dt &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\sqrt{-1} \left( 1 - e^{\sqrt{-1}\frac{\epsilon^2}{4}s} \right)}{se^{\sqrt{-1}\frac{n}{2}s}} e^{\sqrt{-1}s \sum_{j=1}^n \frac{\cos \theta_j}{2}} ds \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\sqrt{-1} \left( 1 - e^{\sqrt{-1}\frac{\epsilon^2}{4}s} \right)}{se^{\sqrt{-1}\frac{n}{2}s}} \left( \prod_{j=1}^n e^{\sqrt{-1}s \frac{\cos \theta_j}{2}} \right) ds \end{aligned}$$

Inserting this formula into (4.39) and performing the integration over  $t$  first, we have

$$\begin{aligned}\text{vol}(B_\epsilon) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\sqrt{-1} \left(1 - e^{\sqrt{-1} \frac{\epsilon^2}{4} s}\right)}{s e^{\sqrt{-1} \frac{n}{2} s}} \left( \int_{[-\pi, \pi]^n} p(\theta_1, \dots, \theta_n) \prod_{j=1}^n e^{\sqrt{-1} s \frac{\cos \theta_j}{2}} d\theta_j \right) \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\sqrt{-1} \left(1 - e^{\sqrt{-1} \frac{\epsilon^2}{4} s}\right)}{s e^{\sqrt{-1} \frac{n}{2} s}} D_n(s) ds,\end{aligned}$$

where

$$D_n(s) = \int_{[-\pi, \pi]^n} p(\theta_1, \dots, \theta_n) \prod_{j=1}^n e^{\sqrt{-1} s \frac{\cos \theta_j}{2}} d\theta_j.$$

Since

$$\prod_{1 \leq i < j \leq n} \left| e^{\sqrt{-1} \theta_i} - e^{\sqrt{-1} \theta_j} \right|^2 = \det \left( e^{\sqrt{-1} (i-1) \theta_k} \right) \overline{\det \left( e^{\sqrt{-1} (i-1) \theta_k} \right)}$$

is a product of two Vandermonde determinants where  $i, k = 1, \dots, n$ . The following fact will be used.

**Proposition 4.1** (Andréief's identity). *For two  $n \times n$  matrices  $M(\mathbf{x})$  and  $N(\mathbf{x})$ , defined by the following:*

$$M(\mathbf{x}) = \begin{bmatrix} M_1(x_1) & M_1(x_2) & \cdots & M_1(x_n) \\ M_2(x_1) & M_2(x_2) & \cdots & M_2(x_n) \\ \vdots & \vdots & \ddots & \vdots \\ M_n(x_1) & M_n(x_2) & \cdots & M_n(x_n) \end{bmatrix}, N(\mathbf{x}) = \begin{bmatrix} N_1(x_1) & N_1(x_2) & \cdots & N_1(x_n) \\ N_2(x_1) & N_2(x_2) & \cdots & N_2(x_n) \\ \vdots & \vdots & \ddots & \vdots \\ N_n(x_1) & N_n(x_2) & \cdots & N_n(x_n) \end{bmatrix}$$

and a function  $w(\cdot)$  such that the integral

$$\int_a^b M_i(x) N_j(x) w(x) dx$$

exists, then the following multiple integral can be evaluated as

$$\int \cdots \int_{\Delta_{a,b}} \det(M(\mathbf{x})) \det(N(\mathbf{x})) \prod_{j=1}^n w(x_j) dx_j = \det \left( \int_a^b M_i(t) N_j(t) w(t) dt \right), \quad (4.40)$$

where  $\Delta_{a,b} := \{\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{R}^n : b \geq x_1 \geq x_2 \geq \cdots \geq x_n \geq a\}$ .

By invoking this identity, we know that

$$\begin{aligned}D_n(s) &= \frac{1}{(2\pi)^n n!} \int \cdots \int_{[-\pi, \pi]^n} \det \left( e^{\sqrt{-1} (i-1) \theta_k} \right) \det \left( e^{-\sqrt{-1} (i-1) \theta_k} \right) \prod_{j=1}^n e^{\sqrt{-1} s \frac{\cos \theta_j}{2}} d\theta_j \\ &= \frac{1}{(2\pi)^n} \det \left( \int_{-\pi}^{\pi} e^{\sqrt{-1} (i-j) \theta} e^{\sqrt{-1} s \frac{\cos \theta}{2}} d\theta \right).\end{aligned}$$

In what follows, we need the Bessel function of the first kind. The definition is

$$J_n(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{\sqrt{-1}(n\theta - x \sin \theta)} d\theta = \sum_{j=0}^{\infty} \frac{(-1)^j}{\Gamma(j+n+1)j!} \left(\frac{x}{2}\right)^{2j+n}.$$

We can list some properties of this Bessel function:

- (i)  $\int_{-\pi}^{\pi} e^{\sqrt{-1}(n\theta + x \sin \theta)} d\theta = \int_{-\pi}^{\pi} e^{\sqrt{-1}(-n\theta - x \sin \theta)} d\theta$
- (ii)  $J_n(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{\sqrt{-1}(n\theta - x \sin \theta)} d\theta = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{\sqrt{-1}(-n\theta + x \sin \theta)} d\theta$
- (iii)  $J_{-n}(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{\sqrt{-1}(-n\theta - x \sin \theta)} d\theta = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{\sqrt{-1}(n\theta + x \sin \theta)} d\theta$

We claim that

$$\int_{-\pi}^{\pi} e^{\sqrt{-1}(n\theta + x \cos \theta)} d\theta = 2\pi e^{\sqrt{-1}\frac{n\pi}{2}} J_n(x). \quad (4.41)$$

Indeed, since

$$\int_{-\pi}^{\pi} e^{\sqrt{-1}(n\theta + x \cos \theta)} d\theta = e^{-\sqrt{-1}\frac{n\pi}{2}} \int_{-\pi}^{\pi} e^{\sqrt{-1}(n\theta + x \sin \theta)} d\theta = e^{-\sqrt{-1}\frac{n\pi}{2}} \cdot 2\pi J_{-n}(x)$$

and  $J_{-n}(x) = (-1)^n J_n(x)$ , the desired claim is obtained.

Thus

$$\begin{aligned} \int_{-\pi}^{\pi} e^{\sqrt{-1}(i-j)\theta} e^{\sqrt{-1}s\frac{\cos \theta}{2}} d\theta &= \int_{-\pi}^{\pi} e^{\sqrt{-1}[(i-j)\theta + \frac{s}{2} \cos \theta]} d\theta \\ &= 2\pi e^{\sqrt{-1}\frac{i-j}{2}\pi} J_{i-j}\left(\frac{s}{2}\right). \end{aligned}$$

We now have

$$D_n(s) = \frac{1}{(2\pi)^n} \det \left( 2\pi e^{\sqrt{-1}\frac{i-j}{2}\pi} J_{i-j}\left(\frac{s}{2}\right) \right) = \frac{1}{(2\pi)^n} \det \left( J_{i-j}\left(\frac{s}{2}\right) \right).$$

By the definition of  $J_n(x)$ , we have  $J_n(-x) = (-1)^n J_n(x)$ , and furthermore,

$$D_n(-s) = D_n(s).$$

Therefore

$$\begin{aligned} \text{vol}(B_\epsilon) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\sqrt{-1} \left(1 - e^{\sqrt{-1} \frac{\epsilon^2}{4} s}\right)}{s e^{\sqrt{-1} \frac{n}{2} s}} D_n(s) ds \\ &= \frac{1}{\pi} \int_0^{\infty} \frac{\sin\left(\frac{ns}{2}\right) + \sin\left(\left(\frac{\epsilon^2}{4} - \frac{n}{2}\right)s\right)}{s} \det\left(J_{i-j}\left(\frac{s}{2}\right)\right) ds, \end{aligned}$$

where we used Euler's formula. In fact, we have

$$\begin{aligned} \frac{1 - e^{\sqrt{-1} \frac{\epsilon^2}{4} s}}{s e^{\sqrt{-1} \frac{n}{2} s}} &= \frac{e^{-\sqrt{-1} \frac{n}{2} s} - e^{\sqrt{-1} \left(\frac{\epsilon^2}{4} - \frac{n}{2}\right) s}}{s} \\ &= \frac{\cos\left(\frac{n}{2}s\right) - \cos\left(\left(\frac{\epsilon^2}{4} - \frac{n}{2}\right)s\right)}{s} + \sqrt{-1} \frac{-\sin\left(\frac{n}{2}s\right) - \sin\left(\left(\frac{\epsilon^2}{4} - \frac{n}{2}\right)s\right)}{s}. \end{aligned}$$

Finally, we get the volume of a metric ball in unitary group  $\mathcal{U}(n)$ :

$$\text{vol}(B_\epsilon) = \frac{1}{\pi} \int_0^{\infty} \frac{\sin\left(\frac{ns}{2}\right) + \sin\left(\left(\frac{\epsilon^2}{4} - \frac{n}{2}\right)s\right)}{s} \det\left(J_{i-j}\left(\frac{s}{2}\right)\right) ds. \quad (4.42)$$

## 5 Appendix I

### 5.1 Matrices with simple eigenvalues form open dense sets of full measure

The present subsection is written based on the Book by Deift and Gioev [4].

Let  $\mathcal{H}_n(\mathbb{C})$  be the set of all  $n \times n$  Hermitian matrices with simple spectrum, i.e. the multiplicity of each eigenvalue are just one. Next we show that  $\mathcal{H}_n(\mathbb{C})$  is an **open and dense set of full measure** (i.e. the Lebesgue measure of the complement is vanished) in  $\mathbf{H}(\mathbb{C}^n)$ , the set of all  $n \times n$  Hermitian matrices.

Assume that  $\tilde{M}$  is an arbitrary Hermitian matrix in  $\mathcal{H}_n(\mathbb{C})$ . with simple spectrum  $\mu_1 < \dots < \mu_n$ , then by standard perturbation theory, all matrices in a neighborhood of  $\tilde{M}$  have simple spectrum. Moreover if  $\tilde{H} \in \mathbf{H}(\mathbb{C}^n)$  with eigenvalues  $\{h_j : j = 1, \dots, n\}$ , then by spectral theorem,

$\tilde{H} = \tilde{U}\Lambda\tilde{U}^*$  for some unitary  $\tilde{U}$  and  $\Lambda = \text{diag}(h_1, \dots, h_n)$ . Now we can always find  $\epsilon_j$  arbitrarily small for all  $j$  so that  $h_j + \epsilon_j$  are distinct for all  $j$ . Thus

$$\tilde{H}_\epsilon := \tilde{U}\text{diag}(h_1 + \epsilon_1, \dots, h_n + \epsilon_n)\tilde{U}^*$$

is a Hermitian matrix with simple spectrum, arbitrarily close to  $\tilde{H}$ . The above two facts show that  $\mathcal{H}_n(\mathbb{C})$  is open and dense. In order to show that  $\mathcal{H}_n(\mathbb{C})$  is of full measure, consider the discriminant:

$$\Delta(\lambda) := \prod_{1 \leq i < j \leq n} (\lambda_j - \lambda_i)^2.$$

By the fundamental theorem of symmetric functions,  $\Delta$  is a polynomial function of the elementary symmetric functions of the  $\lambda_j$ 's, and hence a polynomial function of the  $n^2$  entries  $\tilde{H}_{11}, \dots, \tilde{H}_{nn}, \text{Re}(\tilde{H}_{ij}), \text{Im}(\tilde{H}_{ij}) (i < j)$  of  $\tilde{H}$ . Now if  $\mathcal{H}_n(\mathbb{C})$  were not of full measure, then  $\Delta$  would vanish on a set of positive measure in  $\mathbb{R}^{n^2}$ . It follows that  $\Delta \equiv 0$  because  $\Delta$  is polynomial in  $\mathbb{R}^{n^2}$ .

Let  $H = \text{diag}(1, 2, \dots, n)$  is a Hermitian matrix with distinct spectrum and  $\Delta(H) \neq 0$ . This gives a contradiction and so  $\mathcal{H}_n(\mathbb{C})$  is of full measure in  $\mathbf{H}(\mathbb{C}^n)$ .

Similarly, the set  $\mathcal{S}_n(\mathbb{R})$  of all  $n \times n$  real symmetric matrices with simple spectrum is an [open and dense set of full measure](#) in the set  $\mathbf{S}(\mathbb{R}^n)$  of all real symmetric matrices.

From the above discussion, we conclude that the set of all density matrices with distinct positive eigenvalues is an open and dense set of full measure in the set of all density matrices.

We begin by considering real symmetric matrices of size  $n$ , which is the simplest case. Because  $\mathcal{S}_n(\mathbb{R})$  is of full measure, it is sufficient for the purpose of integration to restrict our attention to matrices  $M \in \mathcal{S}_n(\mathbb{R})$ . Let  $\mu_1 < \dots < \mu_n$  denote the eigenvalues of  $M$ . By spectral theorem,  $M = U\Lambda U^T$ , where  $\Lambda = \text{diag}(\mu_1, \dots, \mu_n)$ . Clearly the columns of  $U$  are defined only up to multiplication by  $\pm 1$ , and so the map  $M \mapsto (\Lambda, U)$ :

$$\mathcal{S}_n(\mathbb{R}) \ni M \mapsto (\Lambda, U) \in \mathbb{R}_+^n \times \mathcal{O}(n)$$

is not well-defined, where

$$\mathbb{R}_+^n = \{(\mu_1, \dots, \mu_n) \in \mathbb{R}^n : \mu_1 < \dots < \mu_n\}, \quad \mathcal{O}(n) = n \times n \text{ orthogonal group.}$$

We consider instead the map

$$\phi_1 : \mathcal{S}_n(\mathbb{R}) \ni M \mapsto (\Lambda, \hat{U}) \in \mathbb{R}_+^n \times (\mathcal{O}(n)/K_1) \tag{5.1}$$

where  $K_1$  is the closed subgroup of  $\mathcal{O}(n)$  containing  $2^n$  elements of the form  $\text{diag}(\pm 1, \dots, \pm 1)$ ,  $\mathcal{O}(n)/K_1$  is the homogeneous manifold obtained by factoring  $\mathcal{O}(n)$  by  $K_1$ , and  $\hat{U} = UK_1$  is the coset containing  $U$ . The map  $\phi_1$  is now clearly well-defined.

The differentiable structure on  $\mathcal{O}(n)/K_1$  is described in the following general result about homogeneous manifolds:

**Proposition 5.1.** *Let  $K$  be a closed subgroup of a Lie group  $G$  and let  $G/K$  be the set  $\{gK : g \in G\}$  of left cosets module  $K$ . Let  $\pi : G \rightarrow G/K$  denote the natural projection  $\pi(g) = gK$ . Then  $G/K$  has a unique manifold structure such that  $\pi$  is  $C^\infty$  and there exist local smooth sections of  $G/K$  in  $G$ , i.e., if  $gK \in G/K$ , there is a neighborhood  $W$  of  $gK$  and a  $C^\infty$  map  $\tau : W \rightarrow G$  such that  $\pi \circ \tau = \text{id}$ .*

In other words, for each  $gK$ , it is possible to choose a  $g' \in gK \subset G$  such that the map  $gK \mapsto g' \equiv \tau(gK)$  is locally defined and smooth.

For example, if  $G = \mathbb{R}^\times$ , the multiplication group of nonzero real numbers, and if  $K$  is the subgroup  $\{\pm 1\}$ , then  $G/K \cong \{x > 0\}$  and  $\pi(a) = |a| \cong \{\pm a\}$ . If  $gK = \{\pm a\}$  for some  $a > 0$ , then  $\tau(\{\pm a\}) = a$ . Also, if  $G = \mathcal{O}(2)$  and  $K_1$  consists of four elements of the form

$$\begin{bmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{bmatrix}, \quad g = \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} \in \mathcal{O}(2),$$

then

$$gK_1 = \left\{ \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix}, \begin{bmatrix} -u_{11} & u_{12} \\ -u_{21} & u_{22} \end{bmatrix}, \begin{bmatrix} u_{11} & -u_{12} \\ u_{21} & -u_{22} \end{bmatrix}, \begin{bmatrix} -u_{11} & -u_{12} \\ -u_{21} & -u_{22} \end{bmatrix} \right\}.$$

Since  $u_{11}^2 + u_{21}^2 = 1 = u_{12}^2 + u_{22}^2$ , each column of  $g$  contains at least one nonzero number: For example, suppose  $u_{21}$  and  $u_{22}$  are nonzero; then there exists a *unique*  $g' \in gK$  such that the elements in the second row of  $g'$  are positive. The same is true for all  $g''K$  close to  $gK$ . Then  $g''K \mapsto g''$  is the desired (local map)  $\tau$ . We can generalize the above construction on  $\mathcal{O}(n)/K_1$ . We will prove the following result:

**Proposition 5.2.**  $\phi_1$  is a diffeomorphism from  $\mathcal{S}_n(\mathbb{R})$  onto  $\mathbb{R}_+^n \times (\mathcal{O}(n)/K_1)$ .

*Proof.* Note that (here and below we always speak of the [real](#) dimensions)

$$\dim \left( \mathbb{R}_+^n \times (\mathcal{O}(n)/K_1) \right) = N + \frac{N(N-1)}{2} = \frac{N(N+1)}{2} \quad (5.2)$$

as it should.

Define the map  $\hat{\phi}_1 : \mathbb{R}_+^n \times (\mathcal{O}(n)/K_1) \rightarrow \mathcal{S}_n(\mathbb{R})$  as follows: If  $(\Lambda, \hat{U})$  lies in  $\mathbb{R}_+^n \times (\mathcal{O}(n)/K_1) \rightarrow \mathcal{S}_n(\mathbb{R})$ , then

$$\hat{\phi}_1(\Lambda, \hat{U}) = U\Lambda U^\top \quad (5.3)$$

where  $U$  is [any](#) matrix in the coset  $\hat{U}$ . If  $U'$  is another such matrix, then  $U' = Uh$  for some  $h \in K_1$  and so

$$U'\Lambda(U')^\top = Uh\Lambda h^\top U^\top = U\Lambda U^\top.$$

Hence  $\hat{\phi}_1$  is well-defined. We will show that

$$\phi_1 \circ \hat{\phi}_1 = \text{id}_{\mathbb{R}_+^n \times (\mathcal{O}(n)/K_1)} \quad (5.4)$$

and

$$\hat{\phi}_1 \circ \phi_1 = \text{id}_{\mathcal{S}_n(\mathbb{R})}. \quad (5.5)$$

Indeed,

$$\phi_1 \left( \hat{\phi}_1 \left( \Lambda, \hat{U} \right) \right) = \phi_1 \left( U \Lambda U^\top \right), \quad U \in \hat{U} \quad (5.6)$$

$$= \left( \Lambda, U K_1 \right) = \left( \Lambda, \hat{U} \right). \quad (5.7)$$

Conversely, if  $M = U \Lambda U^\top \in \mathcal{S}_n(\mathbb{R})$ , then

$$\hat{\phi}_1(\phi_1(M)) = \hat{\phi}_1 \left( \Lambda, \hat{U} = U K_1 \right) = U \Lambda U^\top = M. \quad (5.8)$$

This proves Eq. (5.4) and Eq. (5.5). In order to prove that  $\phi_1$  is a diffeomorphism, it suffices to show that  $\phi_1$  and  $\hat{\phi}_1$  are smooth.

The smoothness of  $\phi_1$  follows from perturbation theory: Fix  $M_0 = U_0 \Lambda_0 U_0^\top \in \mathcal{S}_n(\mathbb{R})$ , where  $\Lambda_0 = \text{diag}(\mu_{01}, \dots, \mu_{0n})$  for  $\mu_{01} < \dots < \mu_{0n}$ , and  $U_0 \in \mathcal{O}(n)$ . Then for  $M$  near  $M_0$ ,  $M \in \mathcal{S}_n(\mathbb{R})$ , the eigenvalues of  $M$ ,  $\mu_1(M) < \dots < \mu_n(M)$ , are smooth functions of  $M$ . Moreover, the associated eigenvectors  $u_j(M)$ ,

$$M u_j(M) = \lambda_j(M) u_j(M), \quad 1 \leq j \leq n,$$

can be chosen orthogonal

$$\langle u_i(M), u_j(M) \rangle = \delta_{ij}, \quad (1 \leq i, j \leq n),$$

and smooth in  $M$ . Indeed, for any  $j$  with  $1 \leq j \leq n$ , let  $P_j$  be the orthogonal projection

$$P_j(M) = \frac{1}{2\pi i} \oint_{\Gamma_j} \frac{1}{s - M} ds \quad (5.9)$$

where  $\Gamma_j$  is a small circle of radius  $\epsilon$  around  $\mu_{0j}$ ,  $|\mu_{0i} - \mu_{0j}| > \epsilon$  for  $i \neq j$ . Then, where  $u_j(M_0)$  is the  $j$ -th column of  $M_0$ ,

$$u_j(M) = \frac{P_j(M) u_j(M_0)}{\sqrt{\langle u_j(M_0), P_j(M) u_j(M_0) \rangle}}, \quad 1 \leq j \leq n,$$

is the desired eigenvector of  $M$ .

Set  $U(M) = [u_1(M), \dots, u_n(M)] \in \mathcal{O}(n)$ . Then  $M \mapsto (\Lambda(M), U(M))$  is smooth and hence

$$\phi_1(M) = (\Lambda(M), U(M)) \equiv \pi(U(M))$$

is smooth, as claimed.

Finally, we show that  $\hat{\phi}_1$  is smooth. Fix  $(\Lambda_0, \hat{U}_0) \in \mathbb{R}_+^n \times (\mathcal{O}(n)/K_1)$  and let  $\tau$  be the lifting map from some neighborhood  $W$  of  $\hat{U}_0$  to  $\mathcal{O}(n)$ . Now for all  $\hat{U} \in W$ ,  $\tau(\hat{U}) \in \hat{U}$  by Proposition 5.1. Hence

$$\hat{\phi}_1(\Lambda, \hat{U}) = \tau(\hat{U})\Lambda\tau(\hat{U})^\top \quad (5.10)$$

from which it is clear that  $\hat{\phi}_1$  is smooth near  $(\Lambda_0, \hat{U}_0)$ , and hence everywhere on  $\mathbb{R}_+^n \times (\mathcal{O}(n)/K_1)$ . This completes the proof.  $\square$

Now consider  $\mathcal{H}_n(\mathbb{C})$ . The calculations are similar to  $\mathcal{S}_n(\mathbb{R})$ . Define the map

$$\phi_2 : \mathcal{H}_n(\mathbb{C}) \ni M \mapsto (\Lambda, \hat{U}) \in \mathbb{R}_+^n \times (\mathcal{U}(n)/K_2). \quad (5.11)$$

Here  $K_2$  is the closed subgroup of  $\mathcal{U}(n)$  given by  $\mathbb{T} \times \dots \times \mathbb{T} = \{\text{diag}(e^{i\theta_1}, \dots, e^{i\theta_n}) : \theta_j \in \mathbb{R}\}$ ,  $\Lambda = \text{diag}(\lambda_1, \dots, \lambda_n)$ ,  $\lambda_1 < \dots < \lambda_n$  as before, and

$$M = U\Lambda U^*, \quad \hat{U} = \pi(U),$$

where  $\pi : \mathcal{U}(n) \rightarrow \mathcal{U}(n)/K_2$  is the natural projection  $U \mapsto \pi(U) = UK_2$  of the unitary group  $\mathcal{U}(n)$  onto the homogeneous manifold  $\mathcal{U}(n)/K_2$ . Let  $\tau : \mathcal{U}(n)/K_2 \rightarrow \mathcal{U}(n)$  denote the (locally defined) lifting map as above. As before,  $\phi_2$  is a well-defined map that is one-to-one from  $\mathcal{H}_n(\mathbb{C})$  to  $\mathcal{U}(n) \rightarrow \mathcal{U}(n)/K_2$ . Note that because  $\dim(\mathcal{U}(n)) = n^2$ ,  $\dim(\mathcal{U}(n)/K_2) = \dim(\mathcal{U}(n)) - \dim(K_2) = n^2 - n$ , as it should.

As before, the proof of the following result is similar to that of Proposition 5.2.

**Proposition 5.3.**  $\phi_2$  is a diffeomorphism from  $\mathcal{H}_n(\mathbb{C})$  onto  $\mathbb{R}_+^n \times (\mathcal{U}(n)/K_2)$ .

## 5.2 Results related to orthogonal groups

**Proposition 5.4.** Let  $Y, X, T \in \mathbb{R}^{n \times n}$ , where  $Y$  and  $T$  are nonsingular,  $X$  is skew symmetric and  $T$  is lower triangular of independent real entries  $y_{ij}$ 's,  $x_{ij}$ 's and  $t_{ij}$ 's respectively. Let  $t_{jj} > 0, j = 1, \dots, n - 1, -\infty < t_{nn} < \infty; -\infty < t_{jk} < \infty, j > k; -\infty < x_{jk} < \infty, j < k$  or  $-\infty < t_{jk} < \infty, j \geq k$  and the first row entries, except the first one, of  $(\mathbb{1}_n + X)^{-1}$  are negative. Then the unique representation  $Y = T[2(\mathbb{1}_n + X)^{-1} - \mathbb{1}_n] = T(\mathbb{1}_n - X)(\mathbb{1}_n + X)^{-1}$  implies that

$$[dY] = 2^{n(n-1)/2} \left( \prod_{j=1}^n |t_{jj}|^{n-j} \right) (\det(\mathbb{1}_n + X))^{-(n-1)} [dT][dX]. \quad (5.12)$$

*Proof.* Take the differentials to get

$$\begin{aligned}
dY &= dT \cdot \left(2(\mathbb{1}_n + X)^{-1} - \mathbb{1}_n\right) + T \cdot \left(-2(\mathbb{1}_n + X)^{-1} \cdot dX \cdot (\mathbb{1}_n + X)^{-1}\right) \\
&= dT \cdot \left(2(\mathbb{1}_n + X)^{-1} - \mathbb{1}_n\right) + T \cdot \left(-\frac{1}{2}(\mathbb{1}_n + Z) \cdot dX \cdot (\mathbb{1}_n + Z)\right) \\
&= dT \cdot Z - \frac{1}{2}T(\mathbb{1}_n + Z) \cdot dX \cdot (\mathbb{1}_n + Z).
\end{aligned}$$

where  $Z = 2(\mathbb{1}_n + X)^{-1} - \mathbb{1}_n$ . Then we have

$$T^{-1} \cdot dY \cdot Z^\top = T^{-1} \cdot dT - \frac{1}{2}(\mathbb{1}_n + Z) \cdot dX \cdot (\mathbb{1}_n + Z^\top).$$

The Jacobian of the transformation of  $T$  and  $X$  going to  $Y$  is equal to the Jacobian of  $dT, dX$  going to  $dY$ . Now treat  $dT, dX, dY$  as variables and everything else as constants. Let

$$dU = T^{-1} \cdot dY \cdot Z^\top, \quad dV = T^{-1} \cdot dT, \quad dW = (\mathbb{1}_n + Z) \cdot dX \cdot (\mathbb{1}_n + Z^\top).$$

Thus

$$\begin{aligned}
[dU] &= \det(T)^{-n} \det(Z^\top)^n [dY] = \det(T)^{-n} [dY] \\
\implies [dY] &= \det(T)^n [dU] = \left(\prod_{j=1}^n |t_{jj}|^n\right) [dU].
\end{aligned}$$

Note that  $\det(Z^\top) = \pm 1$  since  $Z$  is orthogonal. Since  $X$  is skew symmetric, one has

$$[dW] = \det(\mathbb{1}_n + Z)^{n-1} [dX] = 2^{n(n-1)} \det(\mathbb{1}_n + X)^{-(n-1)} [dX].$$

One also has

$$[dV] = \left(\prod_{j=1}^n |t_{jj}|^{-j}\right) [dT].$$

Now we see that

$$dU = dV - \frac{1}{2}dW \implies U = V - \frac{1}{2}W.$$

Let  $U = [u_{ij}], V = [v_{ij}], W = [w_{ij}]$ . Then since  $T$  is lower triangular  $t_{ij} = 0, i < j$  and thus  $V$  is lower triangular, and since  $X$  is skew symmetric  $x_{jj} = 0$  for all  $j$  and  $x_{ij} = -x_{ji}$  for  $i \neq j$  and thus  $W$  is skew symmetric. Thus we have

$$\begin{aligned}
u_{ii} &= v_{ii}, u_{ij} = -\frac{1}{2}w_{ij}, i < j \\
u_{ij} &= v_{ij} + \frac{1}{2}w_{ij}, i > j.
\end{aligned}$$

Take the  $u$ -variables in the order  $u_{ii}, i = 1, \dots, n; u_{ij}, i < j; u_{ij}, i > j$  and  $v_{ii}, i = 1, \dots, n; v_{ij}, i < j; v_{ij}, i > j$ . Then the Jacobian matrix is of the following form:

$$\begin{bmatrix} \mathbb{1}_n & 0 & 0 \\ 0 & (-\frac{1}{2}) \mathbb{1}_{\binom{n}{2}} & 0 \\ 0 & (\frac{1}{2}) \mathbb{1}_{\binom{n}{2}} & \mathbb{1}_{\binom{n}{2}} \end{bmatrix}$$

and the determinant, in absolute value, is  $(\frac{1}{2})^{n(n-1)/2}$ . That is,

$$[dU] = \left(\frac{1}{2}\right)^{n(n-1)/2} [dV][dW].$$

Now substitute for  $[dU]$ ,  $[dW]$  and  $[dV]$ , respectively to obtain the result.  $\square$

**Proposition 5.5.** *Let  $Y, X, D \in \mathbb{R}^{n \times n}$  be of independent real entries, where  $Y$  is symmetric with distinct and nonzero eigenvalues,  $X$  is skew symmetric with the entries of the first row of  $(\mathbb{1}_n + X)^{-1}$ , except the first entry, negative and  $D = \text{diag}(\lambda_1, \dots, \lambda_n)$ , with  $\lambda_1 > \dots > \lambda_n$ . Then, excluding the sign,  $Y = [2(\mathbb{1}_n + X)^{-1} - \mathbb{1}_n] D [2(\mathbb{1}_n + X)^{-1} - \mathbb{1}_n]$  implies that*

$$[dY] = 2^{n(n-1)/2} (\det(\mathbb{1}_n + X))^{-(n-1)} \left( \prod_{i < j} |\lambda_i - \lambda_j| \right) [dX][dD]. \quad (5.13)$$

*Proof.* Let  $Z = 2(\mathbb{1}_n + X)^{-1} - \mathbb{1}_n$ . Take the differentials and reduce to get

$$Z^T \cdot dY \cdot Z = -\frac{1}{2}(\mathbb{1}_n + Z^T) \cdot dX \cdot (\mathbb{1}_n + Z) \cdot D + dD + \frac{1}{2}D \cdot (\mathbb{1}_n + Z^T) \cdot dX \cdot (\mathbb{1}_n + Z).$$

Put

$$dU = Z^T \cdot dY \cdot Z, \quad dW = dD, \quad dV = (\mathbb{1}_n + Z^T) dX (\mathbb{1}_n + Z).$$

Thus

$$dU = -\frac{1}{2}dV \cdot D + \frac{1}{2}D \cdot dV + dW.$$

But since  $dY$  is symmetric and  $Z$  is orthogonal,  $[dU] = \det(Z)^{n+1}[dY] = [dY]$ , excluding the sign. Clearly  $[dW] = [dD]$ . Since  $X$  is skew symmetric we have

$$dV = \det(\mathbb{1}_n + Z)^{n-1}[dX] = 2^{n(n-1)} \det(\mathbb{1}_n + X)^{-(n-1)}[dX].$$

We see that

$$\begin{aligned} du_{ii} &= dw_{ii}, \\ du_{ij} &= \frac{1}{2}(\lambda_i - \lambda_j)dv_{ij}, i < j. \end{aligned}$$

Take the  $u$ -variables in the order  $u_{ii}, i = 1, \dots, n; u_{ij}, i < j$  and the  $w$  and  $v$ -variables in the order  $w_{ii}, i = 1, \dots, n; v_{ij}, i < j$ . Then the matrix of partial derivatives is of the following form:

$$\begin{bmatrix} \mathbb{1} & 0 \\ 0 & M \end{bmatrix},$$

where  $M$  is a diagonal matrix with the diagonal elements  $\frac{1}{2}(\lambda_i - \lambda_j), i < j$ . There are  $n(n-1)/2$  elements. Hence the determinant of the above matrix, in absolute value, is  $2^{-n(n-1)/2} \prod_{i < j} |\lambda_i - \lambda_j|$ . That is,

$$[dU] = 2^{-n(n-1)/2} \left( \prod_{i < j} |\lambda_i - \lambda_j| \right) [dV][dD].$$

Hence

$$[dY] = 2^{n(n-1)/2} \det(\mathbb{1}_n + X)^{-(n-1)} \left( \prod_{i < j} |\lambda_i - \lambda_j| \right) [dX][dD].$$

□

**Remark 5.6.** When integrating over the skew symmetric matrix  $X$  using the transformation in Proposition 5.5, under the unique choice for  $Z = 2(\mathbb{1}_n + X)^{-1} - \mathbb{1}_n$ , observe that

$$2^{n(n-1)/2} \int_X \det(\mathbb{1}_n + X)^{-(n-1)} [dX] = \frac{\pi^{\frac{n^2}{2}}}{\Gamma_n\left(\frac{n}{2}\right)}. \quad (5.14)$$

Note that the  $\lambda_j$ 's are to be integrated out over  $\infty > \lambda_1 > \dots > \lambda_n$  and  $X$  over a unique choice of  $Z$ .

### 5.3 Results related to unitary groups

When  $\tilde{X}$  is skew hermitian, that is,  $\tilde{X}^* = -\tilde{X}$ , it is not difficult to show that  $\mathbb{1} \pm \tilde{X}$  are both nonsingular and  $\tilde{Z} = 2(\mathbb{1} + \tilde{X})^{-1} - \mathbb{1}$  is unitary, that is,  $\tilde{Z}\tilde{Z}^* = \mathbb{1}$ . This property will be made use of in the first result that will be discussed here. Also note that

$$2(\mathbb{1} + \tilde{X})^{-1} - \mathbb{1} = (\mathbb{1} + \tilde{X})^{-1}(\mathbb{1} - \tilde{X}) = (\mathbb{1} - \tilde{X})(\mathbb{1} + \tilde{X})^{-1}.$$

When  $\tilde{X}$  is skew hermitian and of functionally independent complex variables, then there are  $p + 2\frac{n(n-1)}{2} = n^2$  real variables in  $\tilde{X}$ . Let  $\tilde{T}$  be a lower triangular matrix of functionally independent complex variables with the diagonal elements being real. Then there are  $n^2$  real variables in  $\tilde{T}$  also. Thus combined, there are  $2n^2$  real variables in  $\tilde{T}$  and  $\tilde{X}$ . It can be shown that

$$\tilde{Y} = \tilde{T} \left( 2(\mathbb{1} + \tilde{X})^{-1} - \mathbb{1} \right)$$

can produce a one-to-one transformation when the  $t_{jj}$ 's are real and positive.

$$\tilde{Y} = \tilde{T}\tilde{Z},$$

where  $\tilde{Z}\tilde{Z}^* = \mathbb{1}$ ,  $\tilde{T} = [\tilde{t}_{jk}]$ ,  $\tilde{t}_{jj} = t_{jj} > 0, j = 1, \dots, n$ . Then

$$\tilde{Y}\tilde{Y}^* = \tilde{T}\tilde{T}^* \implies t_{11}^2 = \sum_{k=1}^n |\tilde{y}_{1k}|^2.$$

Note that when  $t_{11}$  is real and positive it is uniquely determined in terms of  $\tilde{Y}$ . Now consider the first row elements of  $\tilde{T}\tilde{T}^*$  that is  $t_{11}^2, t_{11}\tilde{t}_{21}, \dots, t_{11}\tilde{t}_{n1}$ . Hence  $\tilde{t}_{21}, \dots, \tilde{t}_{n1}$ , that is, the first column of  $\tilde{T}$  is uniquely determined in terms of  $\tilde{Y}$ . Now consider the second row of  $\tilde{T}\tilde{T}^*$  and so on. Thus  $\tilde{T}$  is uniquely determined in terms of  $\tilde{Y}$ . But  $\tilde{Z} = \tilde{T}^{-1}\tilde{Y}$  and hence  $\tilde{Z}$ , thereby  $\tilde{X}$  is uniquely determined in terms of  $\tilde{Y}$  with no additional restrictions imposed on the elements of  $\tilde{Z}$ .

In this chapter the Jacobians will also be written ignoring the sign as in the previous chapters.

**Proposition 5.7.** *Let  $\tilde{Y}, \tilde{X}$  and  $\tilde{T} = [\tilde{t}_{jk}]$  be  $n \times n$  matrices of functionally independent complex variables where  $\tilde{Y}$  is nonsingular,  $\tilde{X}$  is skew hermitian and  $\tilde{T}$  is lower triangular with real and positive diagonal elements. Ignoring the sign, if*

$$\tilde{Y} = \tilde{T} \left( 2(\tilde{X} + \mathbb{1})^{-1} - \mathbb{1} \right) = \tilde{T}(\mathbb{1} - \tilde{X})(\mathbb{1} + \tilde{X})^{-1},$$

then

$$[d\tilde{Y}] = 2^{n^2} \cdot \left( \prod_{j=1}^n t_{jj}^{2(n-j)+1} \right) \cdot \left| \det((\mathbb{1} + \tilde{X})(\mathbb{1} - \tilde{X})) \right|^{-n} \cdot [d\tilde{X}][d\tilde{T}]. \quad (5.15)$$

*Proof.* Taking differentials in  $\tilde{Y} = \tilde{T} \left( 2(\mathbb{1} + \tilde{X})^{-1} - \mathbb{1} \right)$ , one has

$$d\tilde{Y} = \tilde{T} \left( -2(\mathbb{1} + \tilde{X})^{-1} \cdot d\tilde{X} \cdot (\mathbb{1} + \tilde{X})^{-1} \right) + d\tilde{T} \cdot \left( 2(\mathbb{1} + \tilde{X})^{-1} - \mathbb{1} \right).$$

Let

$$\tilde{Z} = 2(\mathbb{1} + \tilde{X})^{-1} - \mathbb{1} \implies (\mathbb{1} + \tilde{X})^{-1} = \frac{1}{2}(\mathbb{1} + \tilde{Z})$$

and observe that  $\tilde{Z}\tilde{Z}^* = \mathbb{1}$ . Then

$$\tilde{T}^{-1} \cdot d\tilde{Y} \cdot \tilde{Z}^* = -\frac{1}{2}(\mathbb{1} + \tilde{Z}) \cdot d\tilde{X} \cdot (\mathbb{1} + \tilde{Z}^*) + \tilde{T}^{-1} \cdot d\tilde{T}. \quad (5.16)$$

Let  $d\tilde{W} = (\mathbb{1} + \tilde{Z}) \cdot d\tilde{X} \cdot (\mathbb{1} + \tilde{Z}^*)$ . Then

$$[d\tilde{W}] = \left| \det((\mathbb{1} + \tilde{Z})(\mathbb{1} + \tilde{Z}^*)) \right|^n \cdot [d\tilde{X}] = 2^{2n^2} \cdot \left| \det((\mathbb{1} + \tilde{X})(\mathbb{1} - \tilde{X})) \right|^{-n} \cdot [d\tilde{X}]. \quad (5.17)$$

Let  $d\tilde{U} = \tilde{T}^{-1} \cdot d\tilde{T}$ , then

$$[d\tilde{U}] = \left( \prod_{j=1}^n t_{jj}^{-(2j-1)} \right) \cdot [d\tilde{T}]. \quad (5.18)$$

Let  $d\tilde{V} = \tilde{T}^{-1} \cdot d\tilde{Y} \cdot \tilde{Z}^*$ , then

$$[d\tilde{V}] = \left| \det(\tilde{T}\tilde{T}^*) \right|^{-n} \cdot [d\tilde{Y}]. \quad (5.19)$$

Eq. (5.16) reduces to

$$d\tilde{V} = -\frac{1}{2}d\tilde{W} + d\tilde{U}. \quad (5.20)$$

Note that  $d\tilde{W}$  is skew hermitian. Denote

$$\begin{aligned} \tilde{V} &= [\tilde{v}_{jk}] = [v_{jk}^{(1)}] + \sqrt{-1}[v_{jk}^{(2)}], \\ \tilde{W} &= [\tilde{w}_{jk}] = [w_{jk}^{(1)}] + \sqrt{-1}[w_{jk}^{(2)}], \\ \tilde{U} &= [\tilde{u}_{jk}] = [u_{jk}^{(1)}] + \sqrt{-1}[u_{jk}^{(2)}], \end{aligned}$$

where  $v_{jk}^{(m)}, w_{jk}^{(m)}, u_{jk}^{(m)}, m = 1, 2$  are all real. Then from Eq. (5.20), we see that  $\tilde{V} = -\frac{1}{2}\tilde{W} + \tilde{U}$ . Thus

$$\begin{aligned} v_{jk}^{(m)} &= \frac{1}{2}w_{kj}^{(m)} + u_{jk}^{(m)}, j > k, m = 1, 2, \\ v_{jk}^{(m)} &= -\frac{1}{2}w_{jk}^{(m)}, j < k, m = 1, 2, \\ v_{jj}^{(1)} &= u_{jj}^{(1)}, \\ v_{jj}^{(2)} &= -\frac{1}{2}w_{jj}^{(2)}. \end{aligned}$$

The matrices of partial derivatives are the following:

$$\begin{aligned} & \frac{\partial \left( v_{jj}^{(1)}, v_{jj}^{(2)}; v_{jk}^{(1)}, v_{jk}^{(2)} (j < k); v_{jk}^{(1)}, v_{jk}^{(2)} (j > k) \right)}{\partial \left( u_{jj}^{(1)}, w_{jj}^{(2)}; w_{jk}^{(1)}, w_{jk}^{(2)} (j < k); u_{jk}^{(1)}, u_{jk}^{(2)} (j > k) \right)} \\ &= \begin{bmatrix} \mathbb{1}_n & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{2}\mathbb{1}_n & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2}\mathbb{1}_{\binom{n}{2}} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{2}\mathbb{1}_{\binom{n}{2}} & 0 & 0 \\ 0 & 0 & \frac{1}{2}\mathbb{1}_{\binom{n}{2}} & 0 & \mathbb{1}_{\binom{n}{2}} & 0 \\ 0 & 0 & 0 & \frac{1}{2}\mathbb{1}_{\binom{n}{2}} & 0 & \mathbb{1}_{\binom{n}{2}} \end{bmatrix} \end{aligned}$$

where

$$\begin{aligned} A_{11} &= \left( \frac{\partial v_{jj}^{(1)}}{\partial u_{jj}^{(1)}}, j = 1, \dots, n \right) = \mathbb{1}_n, \\ A_{22} &= \left( \frac{\partial v_{jj}^{(2)}}{\partial w_{jj}^{(2)}}, j = 1, \dots, n \right) = -\frac{1}{2} \mathbb{1}_n \end{aligned}$$

and

$$\begin{aligned} A_{33} &= \left( \frac{\partial v_{jk}^{(1)}}{\partial w_{jk}^{(1)}}, j < k \right) = -\frac{1}{2} \mathbb{1}_{\binom{n}{2}}, \\ A_{44} &= \left( \frac{\partial v_{jk}^{(2)}}{\partial w_{jk}^{(2)}}, j < k \right) = -\frac{1}{2} \mathbb{1}_{\binom{n}{2}}, \\ A_{55} &= \left( \frac{\partial v_{jk}^{(1)}}{\partial u_{jk}^{(1)}}, j > k \right) = \mathbb{1}_{\binom{n}{2}}, \\ A_{66} &= \left( \frac{\partial v_{jk}^{(2)}}{\partial u_{jk}^{(2)}}, j > k \right) = \mathbb{1}_{\binom{n}{2}}. \\ A_{53} &= \left( \frac{\partial v_{jk}^{(1)}}{\partial w_{kj}^{(1)}} : j > k \right) = \frac{1}{2} \mathbb{1}_{\binom{n}{2}}, \quad A_{64} = \left( \frac{\partial v_{jk}^{(2)}}{\partial w_{kj}^{(2)}} : j > k \right) = \frac{1}{2} \mathbb{1}_{\binom{n}{2}}. \end{aligned}$$

The determinants of  $A_{11}, \dots, A_{66}$  contribute towards the Jacobian and the product of the determinants, in absolute value, is  $2^{-n(n-1)-n} = 2^{-n^2}$ . Without going through the above procedure one may note from (5.16) that since  $d\tilde{X}$  has  $n^2$  real variables, multiplication by  $\frac{1}{2}$  produces the factor  $2^{-n^2}$  in the Jacobian. From (5.17), (5.18), (5.19) and (5.20) we have

$$\begin{aligned} [d\tilde{Y}] &= \left( \prod_{j=1}^n t_{jj}^{2n} \right) \cdot [d\tilde{V}] = \left( \prod_{j=1}^n t_{jj}^{2n} \right) \cdot 2^{-n^2} \cdot [d\tilde{W}][d\tilde{U}] \\ &= \left( \prod_{j=1}^n t_{jj}^{2n} \right) \cdot 2^{2n^2-n^2} \cdot \left| \det((\mathbb{1} + \tilde{X})(\mathbb{1} - \tilde{X})) \right|^{-n} \cdot \left( \prod_{j=1}^n t_{jj}^{-(2j-1)} \right) \cdot [d\tilde{X}][d\tilde{T}] \\ &= 2^{n^2} \left( \prod_{j=1}^n t_{jj}^{2(n-j)+1} \right) \cdot \left| \det((\mathbb{1} + \tilde{X})(\mathbb{1} - \tilde{X})) \right|^{-n} \cdot [d\tilde{X}][d\tilde{T}]. \end{aligned}$$

This completes the proof. □

**Example 5.8.** Let  $\tilde{X} \in \mathbb{C}^{n \times n}$  skew hermitian matrix of independent complex variables. Then show that

$$\int_{\tilde{X}} [\mathrm{d}\tilde{X}] \left| \det((\mathbb{1} + \tilde{X})(\mathbb{1} - \tilde{X})) \right|^{-n} = \frac{\pi^{n^2}}{2^{n(n-1)} \tilde{\Gamma}_n(n)}.$$

Indeed, let  $\tilde{Y} = [\tilde{y}_{jk}]$  be a  $n \times n$  matrix of independent complex variables. Consider the integral

$$\int_{\tilde{Y}} [\mathrm{d}\tilde{Y}] e^{-\mathrm{Tr}(\tilde{Y}\tilde{Y}^*)} = \prod_{j,k=1}^n \int_{-\infty}^{+\infty} e^{-|\tilde{y}_{jk}|^2} \mathrm{d}\tilde{y}_{jk} = \pi^{n^2}. \quad (5.21)$$

Now consider a transformation  $\tilde{Y} = \tilde{T} \left( 2(\mathbb{1} + \tilde{X})^{-1} - \mathbb{1} \right)$ , where  $\tilde{T} = [\tilde{t}_{jk}]$  is lower triangular with  $t_{jj}$ 's real and positive and no restrictions on  $\tilde{X}$  other than that it is skew hermitian. Then since  $\mathrm{Tr}(\tilde{Y}\tilde{Y}^*) = \mathrm{Tr}(\tilde{T}\tilde{T}^*)$ , from Proposition 5.7 and Eq. (5.21), we have

$$\begin{aligned} \pi^{n^2} &= \int_{\tilde{T}, \tilde{X}} [\mathrm{d}\tilde{X}] [\mathrm{d}\tilde{T}] e^{-\mathrm{Tr}(\tilde{T}\tilde{T}^*)} \cdot 2^{n^2} \cdot \left( \prod_{j=1}^n t_{jj}^{2(n-j)+1} \right) \cdot \left| \det((\mathbb{1} + \tilde{X})(\mathbb{1} - \tilde{X})) \right|^{-n} \\ &= \int_{\tilde{T}} [\mathrm{d}\tilde{T}] e^{-\mathrm{Tr}(\tilde{T}\tilde{T}^*)} \cdot 2^{n^2} \cdot \left( \prod_{j=1}^n t_{jj}^{2(n-j)+1} \right) \times \int_{\tilde{X}} [\mathrm{d}\tilde{X}] \left| \det((\mathbb{1} + \tilde{X})(\mathbb{1} - \tilde{X})) \right|^{-n} \end{aligned}$$

Note that

$$e^{-\mathrm{Tr}(\tilde{T}\tilde{T}^*)} = \exp \left( - \sum_{j=1}^n t_{jj}^2 - \sum_{j>k} |\tilde{t}_{jk}|^2 \right).$$

But

$$\int_{-\infty}^{+\infty} e^{-|\tilde{t}_{jk}|^2} \mathrm{d}\tilde{t}_{jk} = \pi \quad \text{and} \quad \int_0^{+\infty} t_{jj}^{2(n-j)+1} e^{-t_{jj}^2} \mathrm{d}t_{jj} = \frac{1}{2} \Gamma(n-j+1).$$

Hence

$$\int_{\tilde{T}} [\mathrm{d}\tilde{T}] e^{-\mathrm{Tr}(\tilde{T}\tilde{T}^*)} \left( \prod_{j=1}^n t_{jj}^{2(n-j)+1} \right) = 2^{-n} \tilde{\Gamma}_n(n).$$

Substituting this the result follows.

**Remark 5.9.** When a skew hermitian matrix  $\tilde{X}$  is used to parameterize a unitary matrix such as  $\tilde{Z}$  in Proposition 5.7, can we evaluate the Jacobian by direct integration? This will be examined here. Let

$$\mathcal{J}_n := \int_{\tilde{X}} [\mathrm{d}\tilde{X}] \left| \det((\mathbb{1} + \tilde{X})(\mathbb{1} - \tilde{X})) \right|^{-n}.$$

Partition  $\mathbb{1} + \tilde{X}$  as follows:

$$\mathbb{1} + \tilde{X} = \begin{bmatrix} 1 + \tilde{x}_{11} & \tilde{X}_{12} \\ -\tilde{X}_{12}^* & \mathbb{1} + \tilde{X}_1 \end{bmatrix},$$

where  $\tilde{X}_{12}$  represents the first row of  $\mathbb{1} + \tilde{X}$  excluding the first element  $1 + \tilde{x}_{11}$ , and  $\mathbb{1} + \tilde{X}_1$  is obtained from  $\mathbb{1} + \tilde{X}$  by deleting the first row and the first column. Note that

$$\det(\mathbb{1} + \tilde{X}) = \det(\mathbb{1} + \tilde{X}_1) \left( 1 + \tilde{x}_{11} + \tilde{X}_{12}(\mathbb{1} + \tilde{X}_1)^{-1} \tilde{X}_{12}^* \right).$$

Similarly,

$$\mathbb{1} - \tilde{X} = \begin{bmatrix} 1 - \tilde{x}_{11} & -\tilde{X}_{12} \\ \tilde{X}_{12}^* & \mathbb{1} - \tilde{X}_1 \end{bmatrix}$$

and

$$\det(\mathbb{1} - \tilde{X}) = \det(\mathbb{1} - \tilde{X}_1) \left( 1 - \tilde{x}_{11} + \tilde{X}_{12}(\mathbb{1} - \tilde{X}_1)^{-1} \tilde{X}_{12}^* \right).$$

For fixed  $(\mathbb{1} + \tilde{X}_1)$  let  $\tilde{U}_{12} := \tilde{X}_{12}(\mathbb{1} + \tilde{X}_1)^{-1}$ , then

$$[d\tilde{U}_{12}] = \left| \det((\mathbb{1} + \tilde{X}_1)(\mathbb{1} - \tilde{X}_1)) \right|^{-1} \cdot [d\tilde{X}_{12}]$$

and observing that  $\tilde{X}_1^* = -\tilde{X}_1$  we have

$$\tilde{X}_{12}(\mathbb{1} + \tilde{X}_1)^{-1} \tilde{X}_{12}^* = \tilde{U}_{12}(\mathbb{1} - \tilde{X}_1) \tilde{U}_{12}^*.$$

Let  $\tilde{Q}$  be a unitary matrix such that

$$\tilde{Q}^* \tilde{X}_1 \tilde{Q} = \text{diag}(\sqrt{-1}\lambda_1, \dots, \sqrt{-1}\lambda_{n-1})$$

where  $\lambda_1, \dots, \lambda_{n-1}$  are real. Let

$$\tilde{V}_{12} = \tilde{U}_{12} \tilde{Q} = [\tilde{v}_1, \dots, \tilde{v}_{n-1}].$$

Then

$$\tilde{U}_{12}(\mathbb{1} - \tilde{X}_1) \tilde{U}_{12}^* = (1 - \sqrt{-1}\lambda_1) |\tilde{v}_1|^2 + \dots + (1 - \sqrt{-1}\lambda_{n-1}) |\tilde{v}_{n-1}|^2$$

and

$$1 + \tilde{x}_{11} + \tilde{X}_{12}(\mathbb{1} + \tilde{X}_1)^{-1} \tilde{X}_{12}^* = a - \sqrt{-1}b$$

where

$$\begin{aligned} a &= 1 + |\tilde{v}_1|^2 + \dots + |\tilde{v}_{n-1}|^2 \\ b &= -x_{11}^{(2)} + \lambda_1 |\tilde{v}_1|^2 + \dots + \lambda_{n-1} |\tilde{v}_{n-1}|^2 \end{aligned}$$

observing that  $\tilde{x}_{11}$  is purely imaginary, that is,  $\tilde{x}_{11} = \sqrt{-1}x_{11}^{(2)}$ , where  $x_{11}^{(2)}$  is real. Thus

$$\left| \det((\mathbb{1} + \tilde{X})(\mathbb{1} - \tilde{X})) \right|^{-n}$$

yields the factor

$$[(a - \sqrt{-1}b)(a + \sqrt{-1}b)]^{-n} = (a^2 + b^2)^{-n}.$$

So

$$\begin{aligned} \mathcal{J}_n &= \int_{\tilde{X}_1} \int_{\tilde{X}_{12}} \int_{\tilde{x}_{11}} \left( \det(\mathbb{1} + \tilde{X}_1)(a - \sqrt{-1}b) \det(\mathbb{1} - \tilde{X}_1)(a + \sqrt{-1}b) \right)^{-n} [\mathrm{d}\tilde{X}_1][\mathrm{d}\tilde{X}_{12}]\mathrm{d}\tilde{x}_{11} \\ &= \int_{\tilde{X}_1} \int_{\tilde{X}_{12}} \int_{\tilde{x}_{11}} \left| \det((\mathbb{1} + \tilde{X}_1)(\mathbb{1} - \tilde{X}_1)) \right|^{-n} (a^2 + b^2)^{-n} [\mathrm{d}\tilde{X}_1][\mathrm{d}\tilde{X}_{12}]\mathrm{d}\tilde{x}_{11}. \end{aligned}$$

Since

$$[\mathrm{d}\tilde{V}_{12}] = [\mathrm{d}\tilde{U}_{12}] \quad \text{and} \quad [\mathrm{d}\tilde{U}_{12}] = \left| \det((\mathbb{1} + \tilde{X}_1)(\mathbb{1} - \tilde{X}_1)) \right|^{-1} [\mathrm{d}\tilde{X}_{12}],$$

it follows that

$$[\mathrm{d}\tilde{X}_{12}] = \left| \det((\mathbb{1} + \tilde{X}_1)(\mathbb{1} - \tilde{X}_1)) \right| [\mathrm{d}\tilde{V}_{12}].$$

Based on this, we have

$$\mathcal{J}_n = \int_{\tilde{X}_1} \int_{\tilde{V}_{12}} \int_{\tilde{x}_{11}} \left| \det((\mathbb{1} + \tilde{X}_1)(\mathbb{1} - \tilde{X}_1)) \right|^{-(n-1)} (a^2 + b^2)^{-n} [\mathrm{d}\tilde{X}_1][\mathrm{d}\tilde{V}_{12}]\mathrm{d}\tilde{x}_{11}.$$

Then

$$\begin{aligned} \mathcal{J}_n &= \mathcal{J}_{n-1} \int_{\tilde{V}_{12}} \int_{x_{11}^{(2)}} (a^2 + b^2)^{-n} [\mathrm{d}\tilde{V}_{12}] \mathrm{d}x_{11}^{(2)} \\ &= \mathcal{J}_{n-1} \int_{\tilde{V}_{12}} [\mathrm{d}\tilde{V}_{12}] \left( a^{-2n} \int_{x_{11}^{(2)}} \left( 1 + \frac{b^2}{a^2} \right)^{-n} \mathrm{d}x_{11}^{(2)} \right). \end{aligned}$$

Consider the integral over  $x_{11}^{(2)}$ ,  $-\infty < x_{11}^{(2)} < +\infty$ . Change  $x_{11}^{(2)}$  to  $b$  and then to  $c = b/a$ . Then

$$\begin{aligned} \int_{x_{11}^{(2)}} \left( 1 + \frac{b^2}{a^2} \right)^{-n} \mathrm{d}x_{11}^{(2)} &= \int_b \left( 1 + \frac{b^2}{a^2} \right)^{-n} \mathrm{d}b = a \int_{c=-\infty}^{+\infty} (1 + c^2)^{-n} \mathrm{d}c \\ &= 2a \int_0^\infty (1 + c^2)^{-n} \mathrm{d}c = a \frac{\Gamma(\frac{1}{2}) \Gamma(n - \frac{1}{2})}{\Gamma(n)} := k, \end{aligned}$$

by evaluating using a type-2 beta integral after transforming  $u = c^2$ . Hence

$$\begin{aligned} \mathcal{J}_n &= k \mathcal{J}_{n-1} \int_{\tilde{V}_{12}} [\mathrm{d}\tilde{V}_{12}] a^{-(2n-1)} \\ &= k \mathcal{J}_{n-1} \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \left( 1 + |\tilde{v}_1|^2 + \cdots + |\tilde{v}_{n-1}|^2 \right)^{-(2n-1)} \mathrm{d}\tilde{v}_1 \cdots \mathrm{d}\tilde{v}_{n-1}. \end{aligned}$$

For evaluating the integral use the polar coordinates. Let  $\tilde{v}_j = v_j^{(1)} + \sqrt{-1}v_j^{(2)}$ , where  $v_j^{(1)}$  and  $v_j^{(2)}$  are real. Let

$$\begin{cases} v_j^{(1)} = r_j \cos \theta_j, \\ v_j^{(2)} = r_j \sin \theta_j, \end{cases} \quad 0 \leq r_j < \infty, 0 \leq \theta_j \leq 2\pi.$$

Then denoting the multiple integral by  $\mathcal{I}_{n-1}$ , we have

$$\begin{aligned} \mathcal{I}_{n-1} &= \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \left(1 + |\tilde{v}_1|^2 + \cdots + |\tilde{v}_{n-1}|^2\right)^{-(2n-1)} d\tilde{v}_1 \cdots d\tilde{v}_{n-1} \\ &= (2\pi)^{n-1} \int_{r_1=0}^{+\infty} \cdots \int_{r_{n-1}=0}^{+\infty} r_1 \cdots r_{n-1} (1 + r_1^2 + \cdots + r_{n-1}^2)^{-(2n-1)} dr_1 \cdots dr_{n-1}. \end{aligned}$$

Evaluating this by a Dirichlet integral we have

$$\mathcal{I}_{n-1} = \pi^{n-1} \frac{\Gamma(n)}{\Gamma(2n-1)} \text{ for } n \geq 2.$$

Hence for  $n \geq 2$ ,

$$\mathcal{J}_n = \mathcal{J}_{n-1} \pi^{n-1} \sqrt{\pi} \frac{\Gamma(n) \Gamma(n - \frac{1}{2})}{\Gamma(2n-1) \Gamma(n)}.$$

By using the duplication formula for gamma functions

$$\Gamma(2n-1) = \sqrt{\pi} 2^{2n-2} \Gamma\left(n - \frac{1}{2}\right) \Gamma(n).$$

Hence

$$\mathcal{J}_n = \mathcal{J}_{n-1} \frac{\pi^n}{2^{2n-2} \Gamma(n)}.$$

Repeating this process we have

$$\mathcal{J}_n = \frac{\pi^n}{2^{2n-2} \Gamma(n)} \frac{\pi^{n-1}}{2^{2(n-1)-2} \Gamma(n-1)} \cdots \frac{\pi}{2^{2-2} \Gamma(1)} = \frac{\pi^{n^2}}{2^{n(n-1)} \tilde{\Gamma}_n(n)}.$$

This is what we obtained in Example 5.8.

Next we consider a representation of a hermitian positive definite matrix  $\tilde{Y}$  in terms of a skew hermitian matrix  $\tilde{X}$  and a diagonal matrix  $D$  such that

$$\tilde{Y} = \left(2(\mathbb{1} + \tilde{X})^{-1} - \mathbb{1}\right) D \left(2(\mathbb{1} + \tilde{X})^{-1} - \mathbb{1}\right)^*$$

where  $D = \text{diag}(\lambda_1, \dots, \lambda_p)$  with the  $\lambda_j$ 's real distinct and positive, and the first row elements of  $(\mathbb{1} + \tilde{X})^{-1}$  real and of specified signs, which amounts to require  $2(\mathbb{1} + \tilde{X})^{-1} - \mathbb{1} \in \mathcal{U}(n)/\mathcal{U}(1)^{\times n}$ .

In this case it can be shown that the transformation is unique. Note that

$$\tilde{Y} = \tilde{Z} D \tilde{Z}^* = \lambda_1 \tilde{Z}_1 \tilde{Z}_1^* + \cdots + \lambda_p \tilde{Z}_p \tilde{Z}_p^*$$

it indicates that

$$(\tilde{Y} - \lambda_j \mathbb{1}) \tilde{Z}_j = 0, j = 1, \dots, n$$

where  $\tilde{Z}_1, \dots, \tilde{Z}_n$  are the columns of  $\tilde{Z}$  such that  $\langle \tilde{Z}_j, \tilde{Z}_k \rangle = \delta_{jk}$ . Since  $\lambda_1, \dots, \lambda_n$  are the eigenvalues of  $\tilde{Y}$ , which are assumed to be real distinct and positive,  $D$  is uniquely determined in terms of  $\tilde{Y}$ . Note that  $\tilde{Z}_j$  is an eigenvector corresponding to  $\lambda_j$  such that  $\langle \tilde{Z}_j, \tilde{Z}_j \rangle = 1, j = 1, \dots, n$ . Hence  $\tilde{Z}_j$  is uniquely determined in terms of  $\tilde{Y}$  except for a multiple of  $\pm 1, \pm \sqrt{-1}$ . If any particular element of  $\tilde{Z}_j$  is assumed to be real and positive, for example the first element, then  $\tilde{Z}_j$  is uniquely determined. Thus if the first row elements of  $\tilde{Z}$  are real and of specified signs, which is equivalent to saying that the first row elements of  $(\mathbb{1} + \tilde{X})^{-1}$  are real and of specified signs, then the transformation is unique.

**Proposition 5.10.** *Let  $\tilde{Y}$  and  $\tilde{X}$  be  $n \times n$  matrices of functionally independent complex variables such that  $\tilde{Y}$  is hermitian positive definite,  $\tilde{X}$  is skew hermitian and the first row elements of  $(\mathbb{1} + \tilde{X})^{-1}$  are real and of specified signs. Let  $D = \text{diag}(\lambda_1, \dots, \lambda_n)$ , where the  $\lambda_j$ 's are real distinct and positive. Ignoring the sign, if*

$$\tilde{Y} = \left( 2(\tilde{X} + \mathbb{1})^{-1} - \mathbb{1} \right) D \left( 2(\tilde{X} + \mathbb{1})^{-1} - \mathbb{1} \right)^*,$$

then

$$[d\tilde{Y}] = 2^{n(n-1)} \cdot \left( \prod_{j>k} |\lambda_k - \lambda_j|^2 \right) \cdot \left| \det((\mathbb{1} + \tilde{X})(\mathbb{1} - \tilde{X})) \right|^{-n} \cdot [d\tilde{X}][dD].$$

*Proof.* Let  $\tilde{Z} = 2(\mathbb{1} + \tilde{X})^{-1} - \mathbb{1}, \tilde{X}^* = -\tilde{X}$ . Taking the differentials in  $\tilde{Y} = \tilde{Z}D\tilde{Z}^*$  we have

$$d\tilde{Y} = d\tilde{Z} \cdot D \cdot \tilde{Z}^* + \tilde{Z} \cdot dD \cdot \tilde{Z}^* + \tilde{Z} \cdot D \cdot d\tilde{Z}^*. \quad (5.22)$$

But

$$\begin{aligned} d\tilde{Z} &= -2(\mathbb{1} + \tilde{X})^{-1} \cdot d\tilde{X} \cdot (\mathbb{1} + \tilde{X})^{-1} \\ &= -\frac{1}{2}(\mathbb{1} + \tilde{Z}) \cdot d\tilde{X} \cdot (\mathbb{1} + \tilde{Z}) \end{aligned}$$

and

$$\begin{aligned} d\tilde{Z}^* &= 2(\mathbb{1} - \tilde{X})^{-1} \cdot d\tilde{X} \cdot (\mathbb{1} - \tilde{X})^{-1} \\ &= \frac{1}{2}(\mathbb{1} + \tilde{Z}^*) \cdot d\tilde{X} \cdot (\mathbb{1} + \tilde{Z}^*). \end{aligned}$$

From (5.22), one has

$$\tilde{Z}^* \cdot d\tilde{Y} \cdot \tilde{Z} = -\frac{1}{2}(\mathbb{1} + \tilde{Z}^*) \cdot d\tilde{X} \cdot (\mathbb{1} + \tilde{Z}) \cdot D + dD + \frac{1}{2}D \cdot (\mathbb{1} + \tilde{Z}^*) \cdot d\tilde{X} \cdot (\mathbb{1} + \tilde{Z})$$

observing that  $\tilde{Z}^* \tilde{Z} = \mathbb{1}$ . Let

$$d\tilde{U} = \tilde{Z}^* \cdot d\tilde{Y} \cdot \tilde{Z} \implies [d\tilde{U}] = [d\tilde{Y}] \quad \text{since } \tilde{Z}^* \tilde{Z} = \mathbb{1}, \quad (5.23)$$

$$\begin{aligned} d\tilde{V} &= (\mathbb{1} + \tilde{Z}^*) \cdot d\tilde{X} \cdot (\mathbb{1} + \tilde{Z}) = (\mathbb{1} + \tilde{X}^*)^{-1} \cdot 4d\tilde{X} \cdot (\mathbb{1} + \tilde{X})^{-1} \implies \\ [d\tilde{V}] &= 4^{n^2} \cdot \left| \det((\mathbb{1} + \tilde{X})(\mathbb{1} - \tilde{X})) \right|^{-n} \cdot [d\tilde{X}] \end{aligned} \quad (5.24)$$

if there are  $n^2$  free real variables in  $\tilde{X}$ . But in our case there are only  $n^2 - n$  real variables in  $\tilde{X}$  when  $\tilde{X}$  is uniquely chosen and hence

$$[d\tilde{V}] = 4^{n^2-n} \left| \det((\mathbb{1} + \tilde{X})(\mathbb{1} - \tilde{X})) \right|^{-n} \cdot [d\tilde{X}], \quad (5.25)$$

and

$$d\tilde{U} = -\frac{1}{2}d\tilde{V} \cdot D + \frac{1}{2}D \cdot d\tilde{V} + dD. \quad (5.26)$$

From (5.26) and using the fact that  $d\tilde{V}$  is skew hermitian and  $d\tilde{U}$  is hermitian we have

$$du_{jj} = d\lambda_j, \quad du_{jk}^{(m)} = \pm \frac{1}{2}(\lambda_k - \lambda_j) dv_{jk}^{(m)}, j > k, m = 1, 2.$$

Thus the determinant of the Jacobian matrix, in absolute value, is

$$\left( \prod_{j>k} \frac{1}{2} |\lambda_k - \lambda_j| \right)^2 = 2^{-n(n-1)} \prod_{j>k} |\lambda_k - \lambda_j|^2.$$

That is,

$$[d\tilde{U}] = 2^{-n(n-1)} \cdot \prod_{j>k} |\lambda_k - \lambda_j|^2 \cdot [d\tilde{V}][dD].$$

Substituting for  $[d\tilde{U}]$  and  $[d\tilde{V}]$  from (5.23) and (5.24) the result follows.  $\square$

**Example 5.11.** For  $\tilde{X}$  an  $n \times n$  skew hermitian matrix with the first row elements of  $(\mathbb{1} + \tilde{X})^{-1}$  real and of specified signs show that

$$\int_{\tilde{X}} [d\tilde{X}] \left| \det((\mathbb{1} + \tilde{X})(\mathbb{1} - \tilde{X})) \right|^{-n} = \frac{\tilde{\Gamma}_n(n)}{\Delta},$$

where

$$\Delta := 2^{n(n-1)} \int_{\lambda_1 > \dots > \lambda_n > 0} [dD] \left[ \prod_{1 \leq i < j \leq n} |\lambda_i - \lambda_j|^2 \right] e^{-\text{Tr}(D)},$$

with  $D = \text{diag}(\lambda_1, \dots, \lambda_n), \lambda_1 > \dots > \lambda_n > 0$ . Consider a  $n \times n$  hermitian positive definite matrix  $\tilde{Y}$  of functionally independent complex variables. Let

$$\begin{aligned} B &= \int_{\tilde{Y}=\tilde{Y}^*>0} [d\tilde{Y}] e^{-\text{Tr}(\tilde{Y})} = \int_{\tilde{Y}>0} [d\tilde{Y}] \left| \det(\tilde{Y}) \right|^{n-n} e^{-\text{Tr}(\tilde{Y})} \\ &= \tilde{\Gamma}_n(n) = \pi^{\frac{n(n-1)}{2}} \Gamma(n) \Gamma(n-1) \cdots \Gamma(1) \\ &= \pi^{\frac{n(n-1)}{2}} (n-1)! (n-2)! \cdots 1! \end{aligned}$$

evaluating the integral by using a complex matrix-variate gamma integral. Put

$$\tilde{Y} = \tilde{Z} D \tilde{Z}^*, \quad \tilde{Z} = 2(\mathbb{1} + \tilde{X})^{-1} - \mathbb{1}$$

as in Proposition 5.10. Then

$$[d\tilde{Y}] = 2^{n(n-1)} \cdot \left( \prod_{j>k} |\lambda_k - \lambda_j|^2 \right) \cdot \left| \det((\mathbb{1} + \tilde{X})(\mathbb{1} - \tilde{X})) \right|^{-n} \cdot [d\tilde{X}] [dD]$$

and

$$\begin{aligned} B &= \int_{\tilde{X}} [d\tilde{X}] \left| \det((\mathbb{1} + \tilde{X})(\mathbb{1} - \tilde{X})) \right|^{-n} \\ &\quad \times \int_{\lambda_1 > \dots > \lambda_n > 0} [dD] 2^{n(n-1)} \cdot \left( \prod_{j>k} |\lambda_k - \lambda_j|^2 \right) e^{-\text{Tr}(D)}. \end{aligned}$$

Hence the result. From (i) in Example 3.21, we see that

$$\Delta = \left( \frac{2}{\pi} \right)^{n(n-1)} \left( \tilde{\Gamma}_n(n) \right)^2,$$

which implies that, when  $\tilde{X}$  is taken over all skew hermitian under the restriction that the first row elements of  $(\mathbb{1} + \tilde{X})^{-1}$  are real and of specified signs,

$$\int_{\tilde{X}} [d\tilde{X}] \left| \det((\mathbb{1} + \tilde{X})(\mathbb{1} - \tilde{X})) \right|^{-n} = \left( \frac{\pi}{2} \right)^{n(n-1)} \frac{1}{\tilde{\Gamma}_n(n)}.$$

**Remark 5.12.** In fact, we can derive the volume formula (3.36) from Proposition 5.10. The reasoning is as follows:

$$\begin{aligned} \int_{\tilde{Y}>0: \text{Tr}(\tilde{Y})=1} [d\tilde{Y}] &= 2^{n(n-1)} \int_{\lambda_1 > \dots > \lambda_n > 0} [dD] \delta \left( \sum_{j=1}^n \lambda_j - 1 \right) \left( \prod_{j>k} |\lambda_k - \lambda_j|^2 \right) \\ &\quad \times \int_{\tilde{X}} [d\tilde{X}] \left| \det((\mathbb{1} + \tilde{X})(\mathbb{1} - \tilde{X})) \right|^{-n} \\ &= \frac{2^{n(n-1)}}{n!} \times \frac{\Gamma(1) \cdots \Gamma(n) \Gamma(1) \cdots \Gamma(n+1)}{\Gamma(n^2)} \times \left( \frac{\pi}{2} \right)^{n(n-1)} \frac{1}{\tilde{\Gamma}_n(n)} \\ &= \pi^{\frac{n(n-1)}{2}} \frac{\Gamma(1) \cdots \Gamma(n)}{\Gamma(n^2)}. \end{aligned}$$

**Example 5.13.** Let  $\tilde{X} \in \mathbb{C}^{n \times n}$  be a hermitian matrix of independent complex variables. Show that

$$\int_{\tilde{X}} [\mathrm{d}\tilde{X}] e^{-\mathrm{Tr}(\tilde{X}\tilde{X}^*)} = 2^{-\frac{n(n-1)}{2}} \pi^{\frac{n^2}{2}}.$$

Indeed,  $\tilde{X}^* = \tilde{X}$  implies that

$$\mathrm{Tr}(\tilde{X}\tilde{X}^*) = \sum_{j=1}^n x_{jj}^2 + 2 \sum_{i < j} |\tilde{x}_{ij}|^2.$$

Thus

$$\begin{aligned} \int_{\tilde{X}} [\mathrm{d}\tilde{X}] e^{-\mathrm{Tr}(\tilde{X}\tilde{X}^*)} &= \left( \prod_{j=1}^n \int_{-\infty}^{\infty} e^{-x_{jj}^2} \mathrm{d}x_{jj} \right) \times \left( \prod_{i < j} \int e^{-2|\tilde{x}_{ij}|^2} \mathrm{d}\tilde{x}_{ij} \right) \\ &= \pi^{\frac{n}{2}} \times 2^{-\frac{n(n-1)}{2}} \left( \prod_{i < j} \int e^{-|\tilde{x}_{ij}|^2} \mathrm{d}\tilde{x}_{ij} \right) \\ &= \pi^{\frac{n}{2}} \times 2^{-\frac{n(n-1)}{2}} \times \pi^{\frac{n(n-1)}{2}} = 2^{-\frac{n(n-1)}{2}} \pi^{\frac{n^2}{2}}. \end{aligned}$$

**Example 5.14.** By using Example 5.13 or otherwise show that

$$\int_{\infty > \lambda_1 > \dots > \lambda_n > -\infty} \left( \prod_{j > k} |\lambda_k - \lambda_j|^2 \right) \exp \left( - \sum_{j=1}^n \lambda_j^2 \right) \mathrm{d}\lambda_1 \cdots \mathrm{d}\lambda_n = 2^{-\frac{n(n-1)}{2}} \pi^{\frac{n}{2}} \prod_{j=1}^{n-1} j!.$$

Indeed, in Example 5.13 letting  $\tilde{X} = \tilde{U}D\tilde{U}^*$ , where  $D = \mathrm{diag}(\lambda_1, \dots, \lambda_n)$  and  $\tilde{U} \in \mathcal{U}_1(n)$ , gives rise to

$$[\mathrm{d}\tilde{X}] = \left( \prod_{i < j} (\lambda_i - \lambda_j)^2 \right) [\mathrm{d}D][\mathrm{d}\tilde{G}_1], \quad \lambda_1 > \dots > \lambda_n,$$

which means that

$$\begin{aligned} 2^{-\frac{n(n-1)}{2}} \pi^{\frac{n^2}{2}} &= \int_{\tilde{X}} [\mathrm{d}\tilde{X}] e^{-\mathrm{Tr}(\tilde{X}\tilde{X}^*)} \\ &= \int_{\infty > \lambda_1 > \dots > \lambda_n > -\infty} \left( \prod_{i < j} (\lambda_i - \lambda_j)^2 \right) \exp \left( - \sum_{j=1}^n \lambda_j^2 \right) [\mathrm{d}D] \times \int_{\mathcal{U}_1(n)} [\mathrm{d}\tilde{G}_1]. \end{aligned}$$

That is,

$$\int_{\infty > \lambda_1 > \dots > \lambda_n > -\infty} \left( \prod_{i < j} (\lambda_i - \lambda_j)^2 \right) \exp \left( - \sum_{j=1}^n \lambda_j^2 \right) \prod_{j=1}^n \mathrm{d}\lambda_j = 2^{-\frac{n(n-1)}{2}} \pi^{\frac{n}{2}} \prod_{j=1}^{n-1} j!.$$

Therefore

$$\int \left( \prod_{i < j} (\lambda_i - \lambda_j)^2 \right) \exp \left( - \sum_{j=1}^n \lambda_j^2 \right) \prod_{j=1}^n \mathrm{d}\lambda_j = 2^{-\frac{n(n-1)}{2}} \pi^{\frac{n}{2}} \prod_{j=1}^{n-1} j!.$$

## 6 Appendix II: Selberg's integral

This section is rewritten based on Mehta's book [11]. The well-known Selberg's integral is calculated, and some variants and consequences are obtained as well.

**Theorem 6.1** (Selberg's integral). *For any positive integer  $N$ , let  $[dx] = dx_1 \cdots dx_N$ ,*

$$\Delta(x) \equiv \Delta(x_1, \dots, x_N) = \begin{cases} \prod_{1 \leq i < j \leq N} (x_i - x_j), & \text{if } N > 1, \\ 1, & \text{if } N = 1, \end{cases} \quad (6.1)$$

and

$$\Phi(x) \equiv \Phi(x_1, \dots, x_N) = \left( \prod_{j=1}^N x_j^{\alpha-1} (1-x_j)^{\beta-1} \right) |\Delta(x)|^{2\gamma}. \quad (6.2)$$

Then

$$S_N(\alpha, \beta, \gamma) \equiv \int_0^1 \cdots \int_0^1 \Phi(x) [dx] = \prod_{j=0}^{N-1} \frac{\Gamma(\alpha + \gamma j) \Gamma(\beta + \gamma j) \Gamma(\gamma + 1 + \gamma j)}{\Gamma(\alpha + \beta + \gamma(N + j - 1)) \Gamma(1 + \gamma j)}, \quad (6.3)$$

and for  $1 \leq K \leq N$ ,

$$\int_0^1 \cdots \int_0^1 \left( \prod_{j=1}^K x_j \right) \Phi(x) [dx] = \prod_{j=1}^K \frac{\alpha + \gamma(N - j)}{\alpha + \beta + \gamma(2N - j - 1)} \int_0^1 \cdots \int_0^1 \Phi(x) [dx], \quad (6.4)$$

valid for integer  $N$  and complex  $\alpha, \beta, \gamma$  with

$$\operatorname{Re}(\alpha) > 0, \quad \operatorname{Re}(\beta) > 0, \quad \operatorname{Re}(\gamma) > -\min \left( \frac{1}{N}, \frac{\operatorname{Re}(\alpha)}{N-1}, \frac{\operatorname{Re}(\beta)}{N-1} \right). \quad (6.5)$$

*Aomoto's proof.* For brevity, let us write

$$\langle f(x_1, \dots, x_N) \rangle := \frac{\int_{[0,1]^N} f(x_1, \dots, x_N) \Phi(x_1, \dots, x_N) dx_1 \cdots dx_N}{\int_{[0,1]^N} \Phi(x_1, \dots, x_N) dx_1 \cdots dx_N}. \quad (6.6)$$

Firstly, we note that

$$\frac{d}{dx_1} (x_1^a x_2 \cdots x_K \Phi) \quad (6.7)$$

$$= (a + \alpha - 1) x_1^{a-1} x_2 \cdots x_K \Phi - (\beta - 1) \frac{x_1^a x_2 \cdots x_K}{1 - x_1} \Phi + 2\gamma \sum_{j=2}^N \frac{x_1^a x_2 \cdots x_K}{x_1 - x_j} \Phi. \quad (6.8)$$

Indeed,

$$\frac{d}{dx_1} (x_1^a x_2 \cdots x_K \Phi) = a x_1^{a-1} x_2 \cdots x_K \Phi + x_1^a x_2 \cdots x_K \frac{d}{dx_1} \Phi(x_1, \dots, x_N), \quad (6.9)$$

where

$$\frac{d}{dx_1}\Phi = \frac{d}{dx_1} \left( |\Delta(x)|^{2\gamma} \prod_{j=1}^N x_j^{\alpha-1} (1-x_j)^{\beta-1} \right). \quad (6.10)$$

Since

$$[x_1^a x_2 \cdots x_K \Phi(x_1, \dots, x_N)]_{x_1=0}^{x_1=1} = x_2 \cdots x_K \Phi(1, x_2, \dots, x_N) - 0 = 0, \quad (6.11)$$

it follows from integrating between 0 and 1 in (6.7), we get

$$0 = (a + \alpha - 1) \langle x_1^{a-1} x_2 \cdots x_K \rangle - (\beta - 1) \left\langle \frac{x_1^a x_2 \cdots x_K}{1 - x_1} \right\rangle + 2\gamma \sum_{j=2}^N \left\langle \frac{x_1^a x_2 \cdots x_K}{x_1 - x_j} \right\rangle. \quad (6.12)$$

Now for  $a = 1$  and  $a = 2$ , we get

$$0 = \alpha \langle x_2 \cdots x_K \rangle - (\beta - 1) \left\langle \frac{x_1 x_2 \cdots x_K}{1 - x_1} \right\rangle + 2\gamma \sum_{j=2}^N \left\langle \frac{x_1 x_2 \cdots x_K}{x_1 - x_j} \right\rangle, \quad (6.13)$$

$$0 = (\alpha + 1) \langle x_1 x_2 \cdots x_K \rangle - (\beta - 1) \left\langle \frac{x_1^2 x_2 \cdots x_K}{1 - x_1} \right\rangle + 2\gamma \sum_{j=2}^N \left\langle \frac{x_1^2 x_2 \cdots x_K}{x_1 - x_j} \right\rangle. \quad (6.14)$$

Clearly

$$\begin{aligned} \left\langle \frac{x_1 x_2 \cdots x_K}{x_1 - x_j} \right\rangle - \left\langle \frac{x_1^2 x_2 \cdots x_K}{x_1 - x_j} \right\rangle &= \left\langle \frac{x_1 x_2 \cdots x_K}{x_1 - x_j} - \frac{x_1^2 x_2 \cdots x_K}{x_1 - x_j} \right\rangle \\ &= \langle x_1 x_2 \cdots x_K \rangle; \end{aligned} \quad (6.15)$$

interchanging  $x_1$  and  $x_j$  and observing the symmetry,

$$\left\langle \frac{x_1 x_2 \cdots x_K}{x_1 - x_j} \right\rangle = - \left\langle \frac{x_j x_2 \cdots x_K}{x_1 - x_j} \right\rangle = \begin{cases} 0, & \text{if } 2 \leq j \leq K, \\ \frac{1}{2} \langle x_2 \cdots x_K \rangle, & \text{if } K < j \leq N, \end{cases} \quad (6.16)$$

and

$$\left\langle \frac{x_1^2 x_2 \cdots x_K}{x_1 - x_j} \right\rangle = - \left\langle \frac{x_j^2 x_2 \cdots x_K}{x_1 - x_j} \right\rangle = \begin{cases} \frac{1}{2} \langle x_1 x_2 \cdots x_K \rangle, & \text{if } 2 \leq j \leq K, \\ \langle x_1 x_2 \cdots x_K \rangle, & \text{if } K < j \leq N. \end{cases} \quad (6.17)$$

Performing the difference: (6.13) – (6.14) and using the above, we get:

$$(\alpha + 1 + \gamma(K - 1) + 2\gamma(N - K) + \beta - 1) \langle x_1 x_2 \cdots x_K \rangle = (\alpha + \gamma(N - K)) \langle x_2 \cdots x_K \rangle, \quad (6.18)$$

or repeating the process

$$\langle x_1 x_2 \cdots x_K \rangle = \frac{(\alpha + \gamma(N - K))}{\alpha + \beta + \gamma(2N - K - 1)} \langle x_1 \cdots x_{K-1} \rangle = \cdots \quad (6.19)$$

$$= \prod_{j=1}^K \frac{\alpha + \gamma(N - j)}{\alpha + \beta + \gamma(2N - j - 1)}. \quad (6.20)$$

For  $K = N$ , (6.4) can be written as

$$\frac{S_N(\alpha + 1, \beta, \gamma)}{S_N(\alpha, \beta, \gamma)} = \langle x_1 x_2 \cdots x_N \rangle = \prod_{j=1}^N \frac{\alpha + \gamma(N - j)}{\alpha + \beta + \gamma(2N - j - 1)} \quad (6.21)$$

or for a positive integer  $\alpha$ ,

$$\frac{S_N(\alpha, \beta, \gamma)}{S_N(1, \beta, \gamma)} = \frac{S_N(\alpha, \beta, \gamma)}{S_N(\alpha - 1, \beta, \gamma)} \cdots \frac{S_N(2, \beta, \gamma)}{S_N(1, \beta, \gamma)} \quad (6.22)$$

$$= \prod_{j=1}^K \frac{\alpha - 1 + \gamma(N - j)}{\alpha - 1 + \beta + \gamma(2N - j - 1)} \cdots \prod_{j=1}^K \frac{1 + \gamma(N - j)}{1 + \beta + \gamma(2N - j - 1)} \quad (6.23)$$

$$= \prod_{j=1}^K \frac{(\alpha - 1 + \gamma(N - j)) \cdots (1 + \gamma(N - j))}{(\alpha - 1 + \beta + \gamma(2N - j - 1)) \cdots (1 + \beta + \gamma(2N - j - 1))}. \quad (6.24)$$

Note that  $\Gamma(m + \zeta) = (\zeta)_m \Gamma(\zeta)$ , where  $(\zeta)_m := \zeta(\zeta + 1) \cdots (\zeta + m - 1)$ . By using this identity, we get

$$(\alpha - 1 + \gamma(N - j)) \cdots (1 + \gamma(N - j)) = \frac{\Gamma(\alpha + \gamma(N - j))}{\Gamma(1 + \gamma(N - j))}$$

and

$$(\alpha - 1 + \beta + \gamma(2N - j - 1)) \cdots (1 + \beta + \gamma(2N - j - 1)) = \frac{\Gamma(\alpha + \beta + \gamma(2N - j - 1))}{\Gamma(1 + \beta + \gamma(2N - j - 1))}.$$

It follows that

$$\begin{aligned} s_N(\alpha, \beta, \gamma) &= S_N(1, \beta, \gamma) \prod_{j=1}^N \frac{\Gamma(\alpha + \gamma(N - j)) \Gamma(1 + \beta + \gamma(2N - j - 1))}{\Gamma(\alpha + \beta + \gamma(2N - j - 1)) \Gamma(1 + \gamma(N - j))} \\ &= \left( S_N(1, \beta, \gamma) \prod_{j=1}^N \frac{\Gamma(1 + \beta + \gamma(2N - j - 1))}{\Gamma(\beta + \gamma(N - j)) \Gamma(1 + \gamma(N - j))} \right) \prod_{j=1}^N \frac{\Gamma(\alpha + \gamma(N - j)) \Gamma(\beta + \gamma(N - j))}{\Gamma(\alpha + \beta + \gamma(2N - j - 1))}. \end{aligned}$$

As  $S_N(\alpha, \beta, \gamma) = S_N(\beta, \alpha, \gamma)$  by the fact that  $\Delta(1 - x) = \pm \Delta(x)$ , that is,  $S_N(\alpha, \beta, \gamma)$  is a symmetric function of  $\alpha$  and  $\beta$ , at the same time, the following factor is also symmetric in  $\alpha$  and  $\beta$ :

$$\prod_{j=1}^N \frac{\Gamma(\alpha + \gamma(N - j)) \Gamma(\beta + \gamma(N - j))}{\Gamma(\alpha + \beta + \gamma(2N - j - 1))}.$$

It follows that the factor

$$S_N(1, \beta, \gamma) \prod_{j=1}^N \frac{\Gamma(1 + \beta + \gamma(2N - j - 1))}{\Gamma(\beta + \gamma(N - j)) \Gamma(1 + \gamma(N - j))}$$

should be a symmetric function of  $\alpha$  and  $\beta$ . But, however, this factor is independent of  $\alpha$ , therefore it should be also independent of  $\beta$  by the symmetry. Denote this factor by  $c(\gamma, N)$ , we get

$$\begin{aligned} S_N(\alpha, \beta, \gamma) &= c(\gamma, N) \prod_{j=1}^N \frac{\Gamma(\alpha + \gamma(N - j)) \Gamma(\beta + \gamma(N - j))}{\Gamma(\alpha + \beta + \gamma(2N - j - 1))} \\ &= c(\gamma, N) \prod_{j=1}^N \frac{\Gamma(\alpha + \gamma(j - 1)) \Gamma(\beta + \gamma(j - 1))}{\Gamma(\alpha + \beta + \gamma(N + j - 2))} \end{aligned} \quad (6.25)$$

where  $c(\gamma, N)$  is independent of  $\alpha$  and  $\beta$ .

To determine  $c(\gamma, N)$ , put  $\alpha = \beta = 1$ ;

$$S_N(1, 1, \gamma) = \int_{[0,1]^N} |\Delta(x)|^{2\gamma} dx = c(\gamma, N) \prod_{j=1}^N \frac{\Gamma(1 + \gamma(j-1))^2}{\Gamma(2 + \gamma(N+j-2))}. \quad (6.26)$$

Let  $y$  be the largest of the  $x_1, \dots, x_N$  and replace the other  $x_j$  by  $x_j = yt_j$ , where  $0 \leq t_j \leq 1$ . Without loss of generality, we assume that  $y = x_N$ . Then  $x_j = yt_j$  for  $j = 1, \dots, N-1$ . Then

$$|\Delta(x_1, \dots, x_N)|^{2\gamma} = y^{\gamma N(N-1)} \cdot |\Delta(t_1, \dots, t_{N-1})|^{2\gamma} \cdot \prod_{j=1}^{N-1} (1-t_j)^{2\gamma}, \quad (6.27)$$

and the Jacobian of change of variables is

$$\left| \det \left( \frac{\partial(x_1, \dots, x_N)}{\partial(y, t_1, \dots, t_{N-1})} \right) \right| = \left| \begin{vmatrix} t_1 & t_2 & \cdots & t_{N-1} & 1 \\ y & 0 & 0 & \cdots & 0 \\ 0 & y & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & y & 0 \end{vmatrix} \right| = y^{N-1}. \quad (6.28)$$

Now we have

$$\begin{aligned} S_N(1, 1, \gamma) &= N! \int_{0 \leq x_1 \leq \dots \leq x_N \leq 1} |\Delta(x)|^{2\gamma} dx \\ &= N! \int_0^1 y^{\gamma N(N-1)} y^{N-1} dy \cdot \int_{0 \leq t_1 \leq \dots \leq t_{N-1} \leq 1} |\Delta(t_1, \dots, t_{N-1})|^{2\gamma} \prod_{j=1}^{N-1} (1-t_j)^{2\gamma} dt_1 \dots dt_{N-1} \\ &= \left( N \int_0^1 y^{\gamma N(N-1)} y^{N-1} dy \right) \left( \int_{[0,1]^{N-1}} |\Delta(t_1, \dots, t_{N-1})|^{2\gamma} \prod_{j=1}^{N-1} (1-t_j)^{2\gamma} dt_1 \dots dt_{N-1} \right) \\ &= \frac{1}{\gamma(N-1) + 1} S_{N-1}(1, 2\gamma + 1, \gamma), \end{aligned}$$

that is

$$S_N(1, 1, \gamma) = \frac{1}{\gamma(N-1) + 1} S_{N-1}(1, 2\gamma + 1, \gamma) \quad (6.29)$$

$$= \frac{c(\gamma, N-1)}{\gamma(N-1) + 1} \cdot \prod_{j=1}^{N-1} \frac{\Gamma(1 + \gamma(j-1))\Gamma(2\gamma + 1 + \gamma(j-1))}{\Gamma(1 + 2\gamma + 1 + \gamma(N-1+j-2))} \quad (6.30)$$

$$= \frac{c(\gamma, N-1)}{\gamma(N-1) + 1} \cdot \prod_{j=1}^{N-1} \frac{\Gamma(1 + \gamma(j-1))\Gamma(1 + \gamma + \gamma j)}{\Gamma(2 + \gamma(N+j-1))}. \quad (6.31)$$

Thus

$$c(\gamma, N) \prod_{j=1}^N \frac{\Gamma(1 + \gamma(j-1))^2}{\Gamma(2 + \gamma(N+j-2))} = S_N(1, 1, \gamma) = \frac{c(\gamma, N-1)}{\gamma(N-1) + 1} \cdot \prod_{j=1}^{N-1} \frac{\Gamma(1 + \gamma(j-1))\Gamma(1 + \gamma + \gamma j)}{\Gamma(2 + \gamma(N+j-1))}.$$

This implies that

$$\frac{c(\gamma, N)}{c(\gamma, N-1)} = \frac{\Gamma(1+\gamma N)}{\Gamma(1+\gamma)}, \quad (6.32)$$

or

$$c(\gamma, N) = c(\gamma, 1) \cdot \left( \frac{c(\gamma, N)}{c(\gamma, N-1)} \cdots \frac{c(\gamma, 2)}{c(\gamma, 1)} \right) = c(\gamma, 1) \cdot \left( \frac{\Gamma(1+\gamma N)}{\Gamma(1+\gamma)} \cdots \frac{\Gamma(1+\gamma 2)}{\Gamma(1+\gamma)} \right). \quad (6.33)$$

Finally we get

$$c(\gamma, N) = c(\gamma, 1) \cdot \prod_{j=2}^N \frac{\Gamma(1+\gamma j)}{\Gamma(1+\gamma)} = \prod_{j=2}^N \frac{\Gamma(1+\gamma j)}{\Gamma(1+\gamma)}, \quad (6.34)$$

where the fact that  $c(\gamma, 1) = 1$  is trivial.  $\square$

**Remark 6.2.** Note that the conclusion is derived here for integers  $\alpha, \beta$  and complex  $\gamma$ .

A slight change of reasoning due to Askey gives (6.25) directly for complex  $\alpha, \beta$  and  $\gamma$  as follows. (6.88) and the symmetry identity that  $S_N(\alpha, \beta, \gamma) = S_N(\beta, \alpha, \gamma)$  give the ratio of  $S_N(\alpha, \beta, \gamma)$  and  $S_N(\alpha, \beta + m, \gamma)$  for any integer  $m$ ,

$$\frac{S_N(\alpha, \beta + m, \gamma)}{S_N(\alpha, \beta, \gamma)} = \frac{S_N(\alpha, \beta + m, \gamma)}{S_N(\alpha, \beta + m - 1, \gamma)} \cdots \frac{S_N(\alpha, \beta + 1, \gamma)}{S_N(\alpha, \beta, \gamma)} \quad (6.35)$$

$$= \prod_{j=1}^N \frac{(\beta + m - 1) + \gamma(N - j)}{\alpha + (\beta + m - 1) + \gamma(2N - j - 1)} \cdots \prod_{j=1}^N \frac{\beta + \gamma(N - j)}{\alpha + \beta + \gamma(2N - j - 1)} \quad (6.36)$$

$$= \prod_{j=1}^N \frac{(\beta + \gamma(N - j))_m}{(\alpha + \beta + \gamma(2N - j - 1))_m}, \quad (6.37)$$

where we have used the notation

$$(a)_m = \frac{\Gamma(a + m)}{\Gamma(a)}; \quad m \geq 0;$$

i.e.  $(a)_0 = 1$  and  $(a)_m := a(a + 1) \cdots (a + m - 1)$  for  $m \geq 1$ . Now

$$S_N(\alpha, \beta + m, \gamma) = \int_{[0,1]^N} |\Delta(x)|^{2\gamma} \prod_{j=1}^N x_j^{\alpha-1} (1 - x_j)^{\beta+m-1} dx_j \quad (6.38)$$

$$= m^{-\alpha N - \gamma N(N-1)} \int_{[0,m]^{-N}} |\Delta(x)|^{2\gamma} \prod_{j=1}^N x_j^{\alpha-1} \left(1 - \frac{x_j}{m}\right)^{\beta+m-1} dx_j. \quad (6.39)$$

Thus

$$S_N(\alpha, \beta, \gamma) = S_N(\alpha, \beta + m, \gamma) \cdot \prod_{j=1}^N \frac{(\alpha + \beta + \gamma(2N - j - 1))_m}{(\beta + \gamma(N - j))_m} \quad (6.40)$$

$$= m^{-\alpha N - \gamma N(N-1)} \int_{[0,m]^{-N}} |\Delta(x)|^{2\gamma} \prod_{j=1}^N x_j^{\alpha-1} \left(1 - \frac{x_j}{m}\right)^{\beta+m-1} dx_j \prod_{j=1}^N \frac{(\alpha + \beta + \gamma(2N - j - 1))_m}{(\beta + \gamma(N - j))_m} \quad (6.41)$$

$$= \prod_{j=1}^N \left( \frac{(\alpha + \beta + \gamma(2N - j - 1))_m}{(\beta + \gamma(N - j))_m} m^{-\alpha - \gamma(N-1)} \right) \int_{[0,m]^{-N}} |\Delta(x)|^{2\gamma} \prod_{j=1}^N x_j^{\alpha-1} \left(1 - \frac{x_j}{m}\right)^{\beta+m-1} dx_j \quad (6.42)$$

Denote  $a_j = \alpha + \gamma(N - j)$ ,  $b_j = \beta + \gamma(N - j)$  and  $c_j = \alpha + \beta + \gamma(2N - j - 1)$ , we have

$$\frac{(c)_m}{(b)_m} m^{b-c} = \frac{\Gamma(b)}{\Gamma(c)} \frac{\Gamma(c+m)}{\Gamma(b+m)} m^{b-c} = \frac{\Gamma(b)}{\Gamma(c)} \frac{\frac{\Gamma(c+m)}{\Gamma(m)m^c}}{\frac{\Gamma(b+m)}{\Gamma(m)m^b}}.$$

By using the fact that

$$\lim_{m \rightarrow \infty} \frac{\Gamma(m+c)}{\Gamma(m)m^c} = 1 \quad (\forall c \in \mathbb{R}),$$

it follows that

$$\lim_{m \rightarrow \infty} \frac{(c)_m}{(b)_m} m^{b-c} = \frac{\Gamma(b)}{\Gamma(c)}, \quad (6.43)$$

therefore

$$\lim_{m \rightarrow \infty} \prod_{j=1}^N \frac{(c_j)_m}{(b_j)_m} m^{b_j-c_j} = \prod_{j=1}^N \frac{\Gamma(b_j)}{\Gamma(c_j)}. \quad (6.44)$$

Taking  $m \rightarrow \infty$  in (6.40) gives rise to the following:

$$S_N(\alpha, \beta, \gamma) = \left( \prod_{j=1}^N \frac{\Gamma(b_j)}{\Gamma(c_j)} \right) \int_0^\infty \cdots \int_0^\infty |\Delta(x)|^{2\gamma} \prod_{j=1}^N x_j^{\alpha-1} \exp(-x_j) dx_j. \quad (6.45)$$

Furthermore,

$$S_N(\alpha, \beta, \gamma) = \left( \prod_{j=1}^N \frac{\Gamma(a_j)\Gamma(b_j)}{\Gamma(c_j)} \right) \frac{\int_0^\infty \cdots \int_0^\infty |\Delta(x)|^{2\gamma} \prod_{j=1}^N x_j^{\alpha-1} \exp(-x_j) dx_j}{\prod_{j=1}^N \Gamma(a_j)}. \quad (6.46)$$

By the symmetry of  $\alpha$  and  $\beta$ , it follows that the factor

$$\frac{\int_0^\infty \cdots \int_0^\infty |\Delta(x)|^{2\gamma} \prod_{j=1}^N x_j^{\alpha-1} \exp(-x_j) dx_j}{\prod_{j=1}^N \Gamma(a_j)} \quad (6.47)$$

is a symmetric function of  $\alpha$  and  $\beta$ . Since it is independent of  $\beta$ , it is also independent of  $\alpha$  by the symmetry of  $\alpha$  and  $\beta$ . Thus

$$\frac{\int_0^\infty \cdots \int_0^\infty |\Delta(x)|^{2\gamma} \prod_{j=1}^N x_j^{\alpha-1} \exp(-x_j) dx_j}{\prod_{j=1}^N \Gamma(a_j)} = \frac{\int_0^\infty \cdots \int_0^\infty |\Delta(x)|^{2\gamma} \prod_{j=1}^N x_j^{\beta-1} \exp(-x_j) dx_j}{\prod_{j=1}^N \Gamma(b_j)} \quad (6.48)$$

which is denoted by  $c(\gamma, N)$ . This indicates that

$$\int_0^\infty \cdots \int_0^\infty |\Delta(x)|^{2\gamma} \prod_{j=1}^N x_j^{\alpha-1} \exp(-x_j) dx_j = c(\gamma, N) \prod_{j=1}^N \Gamma(a_j) \quad (6.49)$$

$$= \prod_{j=2}^N \frac{\Gamma(1+\gamma j)}{\Gamma(1+\gamma)} \prod_{j=1}^N \Gamma(a_j) = \prod_{j=1}^N \frac{\Gamma(\alpha + \gamma(j-1))\Gamma(1+\gamma j)}{\Gamma(1+\gamma)}. \quad (6.50)$$

**Remark 6.3.** Now we show that

$$\lim_{n \rightarrow \infty} \frac{\Gamma(n + \alpha)}{\Gamma(n)n^\alpha} = 1 \quad (\forall \alpha \in \mathbb{R}^+ \cup \{0\}).$$

Indeed, in order to prove this fact, we need a limit representation of the gamma function given by Carl Friedrich Gauss via Euler's representation of  $n! = \prod_{k=1}^{\infty} \frac{(1+\frac{1}{k})^n}{1+\frac{n}{k}}$ :

$$\Gamma(z) = \lim_{n \rightarrow \infty} \frac{n!n^z}{z(z+1) \cdots (z+n)}.$$

$$\lim_{n \rightarrow \infty} \frac{\Gamma(n + \alpha)}{\Gamma(n)n^\alpha} = \lim_{n \rightarrow \infty} \frac{\Gamma(\alpha)\alpha(\alpha+1) \cdots (\alpha+n-1)}{(n-1)!n^\alpha} \quad (6.51)$$

$$= \Gamma(\alpha) \lim_{n \rightarrow \infty} \frac{\alpha(\alpha+1) \cdots (\alpha+n)}{n!n^\alpha} \frac{n}{n+\alpha} \quad (6.52)$$

$$= \Gamma(\alpha) \lim_{n \rightarrow \infty} \frac{\alpha(\alpha+1) \cdots (\alpha+n)}{n!n^\alpha} \lim_{n \rightarrow \infty} \frac{n}{n+\alpha} \quad (6.53)$$

$$= \Gamma(\alpha) \cdot \frac{1}{\Gamma(\alpha)} \cdot 1 = 1. \quad (6.54)$$

Another short and elementary proof of this fact can be derived from a result related to inequalities for Gamma function ratios [18]:

$$x(x+a)^{a-1} \leq \frac{\Gamma(x+a)}{\Gamma(x)} \leq x^a \quad (\forall a \in [0, 1]). \quad (6.55)$$

Indeed,

$$\frac{\Gamma(x+\alpha)}{\Gamma(x)} \sim x^\alpha \quad (6.56)$$

as  $x \rightarrow \infty$  with  $\alpha$  fixed. (This was the objective of Wendel's article.) To show this, first suppose that  $\alpha \in [0, 1]$ . Then Eq. (6.55) gives

$$\left(1 + \frac{\alpha}{x}\right)^{\alpha-1} \leq \frac{\Gamma(x+\alpha)}{\Gamma(x)x^\alpha} \leq 1,$$

leading to  $\lim_{x \rightarrow \infty} \frac{\Gamma(x+\alpha)}{\Gamma(x)x^\alpha} = 1$  since  $\lim_{x \rightarrow \infty} \left(1 + \frac{\alpha}{x}\right)^{\alpha-1} = \lim_{x \rightarrow \infty} 1 = 1$ . For  $\alpha > 1$ , the statement now follows from the fact that

$$\frac{\Gamma(x+\alpha)}{\Gamma(x)} = (x+\alpha-1) \frac{\Gamma(x+\alpha-1)}{\Gamma(x)}.$$

We are done.

**Corollary 6.4** (Laguerre's integral). *By letting  $x_j = y_j/L$  and  $\beta = L+1$  in Selberg's integral and taking the limit  $L \rightarrow \infty$ , we obtain*

$$\int_0^\infty \cdots \int_0^\infty |\Delta(x)|^{2\gamma} \prod_{j=1}^N x_j^{\alpha-1} \exp(-x_j) dx_j = \prod_{j=1}^N \frac{\Gamma(\alpha + \gamma(j-1))\Gamma(1+\gamma j)}{\Gamma(1+\gamma)}. \quad (6.57)$$

**Corollary 6.5** (Hermite's integral). *By letting  $x_j = y_j/L$  and  $\alpha = \beta = \lambda L^2 + 1$  in Selberg's integral and taking the limit  $L \rightarrow \infty$ , we obtain*

$$\int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} |\Delta(x)|^{2\gamma} \prod_{j=1}^N \exp(-\lambda x_j^2) dx_j = (2\pi)^{N/2} (2\lambda)^{-N(\gamma(N-1)+1)/2} \prod_{j=1}^N \frac{\Gamma(1+\gamma j)}{\Gamma(1+\gamma)}. \quad (6.58)$$

**Remark 6.6.** For an integer  $\gamma$ , the last equation can also be written as a finite algebraic identity. Firstly, we note that

$$\int_{-\infty}^{+\infty} \exp(-a^2 x^2 - 2iax\lambda) x^n dx = \left( \frac{i}{2a} \frac{d}{d\lambda} \right)^n \int_{-\infty}^{+\infty} \exp(-a^2 x^2 - 2iax\lambda) dx \quad (6.59)$$

$$= \left( \frac{i}{2a} \frac{d}{d\lambda} \right)^n \left( \exp(-\lambda^2) \int_{-\infty}^{+\infty} \exp(-(ax + i\lambda)^2) dx \right) \quad (6.60)$$

$$= \frac{\sqrt{\pi}}{a} \left( \frac{i}{2a} \frac{d}{d\lambda} \right)^n \exp(-\lambda^2), \quad (6.61)$$

letting  $\lambda = 0$  in the above reasoning, we get

$$\int_{-\infty}^{+\infty} \exp(-a^2 x^2) x^n dx = \frac{\sqrt{\pi}}{a} \left( \frac{i}{2a} \frac{d}{d\lambda} \right)^n \exp(-\lambda^2) \Big|_{\lambda=0} = \frac{\sqrt{\pi}}{a} \left( \frac{i}{2a} \frac{d}{dx} \right)^n \exp(-x^2) \Big|_{x=0}, \quad (6.62)$$

we replace  $a$  by  $\sqrt{a}$  in the last equation, we get

$$\int_{-\infty}^{+\infty} \exp(-ax^2) x^n dx = \sqrt{\frac{\pi}{a}} \left( \frac{i}{2\sqrt{a}} \frac{d}{dx} \right)^n \exp(-x^2) \Big|_{x=0}, \quad (6.63)$$

thus (6.58) therefore takes the form:

$$\left( \frac{i}{2\sqrt{a}} \right)^{\gamma N(N-1)} \prod_{1 \leq p < q \leq N} \left( \frac{\partial}{\partial x_p} - \frac{\partial}{\partial x_q} \right)^{2\gamma} \exp \left( - \sum_{j=1}^N x_j^2 \right) \Big|_{(x_1, \dots, x_N)=0} = (2a)^{-\gamma N(N-1)/2} \prod_{j=1}^N \frac{(j\gamma)!}{\gamma!} \quad (6.64)$$

Replacing the exponential by its power series expansion, one notes that the term  $(-\sum_{j=1}^N x_j^2)^\ell$  gives zero on differentiation if  $\ell < \gamma N(N-1)/2$ , and leaves a homogeneous polynomial of order  $\ell - \gamma N(N-1)/2$  in the variables  $x_1, \dots, x_N$ , if  $\ell > \gamma N(N-1)/2$ . On setting  $x_j = 0, j = 1, \dots, N$ , one sees that therefore that there is only one term, corresponding to  $\ell = \gamma N(N-1)/2$ , which gives a non-zero contribution. So

$$\prod_{1 \leq p < q \leq N} \left( \frac{\partial}{\partial x_p} - \frac{\partial}{\partial x_q} \right)^{2\gamma} \left( \sum_{j=1}^N x_j^2 \right)^\ell = 2^\ell \ell! \prod_{j=1}^N \frac{(j\gamma)!}{\gamma!}, \quad (6.65)$$

where  $\ell = \gamma N(N-1)/2$ . If  $P(x) := P(x_1, \dots, x_N)$  and  $Q(x) := Q(x_1, \dots, x_N)$  are homogeneous polynomials in  $x := (x_1, \dots, x_N)$  of the same degree, then a little reflection shows that  $P(\partial/\partial x)Q(x)$  is a constant which is also equal to  $Q(\partial/\partial x)P(x)$ . Thus one can interchange the roles of  $x_j$  and  $\partial/\partial x_j$  to get

$$\left( \sum_{k=1}^N \frac{\partial^2}{\partial x_k^2} \right)^\ell \prod_{1 \leq i < j \leq N} (x_i - x_j)^{2\gamma} = 2^\ell \ell! \prod_{j=1}^N \frac{(j\gamma)!}{\gamma!}, \quad (6.66)$$

where  $\ell = \gamma N(N-1)/2$ .

**Corollary 6.7.** *It holds that*

$$\int_{[0,2\pi]^N} \left| \Delta(e^{i\theta_1}, \dots, e^{i\theta_N}) \right|^{2\gamma} \prod_{k=1}^N \frac{d\theta_k}{2\pi} = \int_0^{2\pi} \dots \int_0^{2\pi} \prod_{1 \leq i < j \leq N} \left| e^{i\theta_i} - e^{i\theta_j} \right|^{2\gamma} \prod_{k=1}^N \frac{d\theta_k}{2\pi} \quad (6.67)$$

$$= \frac{(N\gamma)!}{(\gamma!)^N}, \quad (6.68)$$

where  $\gamma$  is non-negative integer.

**Corollary 6.8.** *It holds that when there is no overlap in the two sets of factors, the result is*

$$\mathcal{B}(K_1, K_2) = \int_0^1 \dots \int_0^1 \prod_{i=1}^{K_1} x_i \prod_{j=K_1+1}^{K_1+K_2} (1-x_j) \Phi(x) dx \quad (6.69)$$

$$= \mathcal{I}_N(\alpha, \beta, \gamma) \frac{\prod_{i=1}^{K_1} (\alpha + \gamma(N-i)) \prod_{j=1}^{K_2} (\beta + \gamma(N-j))}{\prod_{k=1}^{K_1+K_2} (\alpha + \beta + \gamma(2N-k-1))}, \quad (6.70)$$

where  $K_1, K_2 \geq 0, K_1 + K_2 \leq N$ , and when there is overlap

$$\mathcal{C}(K_1, K_2, K_3) = \int_0^1 \dots \int_0^1 \prod_{i=1}^{K_1} x_i \prod_{j=K_1+1-K_3}^{K_1+K_2-K_3} (1-x_j) \Phi(x) dx \quad (6.71)$$

$$= \mathcal{B}(K_1, K_2) \prod_{k=1}^{K_3} \frac{\alpha + \beta + \gamma(N-k-1)}{\alpha + \beta + 1 + \gamma(2N-k-1)}, \quad (6.72)$$

where  $K_1, K_2, K_3 \geq 0, K_1 + K_2 - K_3 \leq N$ .

**Remark 6.9.** Still another integral of interest is the average value of the product of traces of the matrix in the circular ensembles. For example,

$$S_N(p, \gamma) := \frac{1}{(2\pi)^N} \frac{(\gamma!)^N}{(N\gamma)!} \int_0^{2\pi} \dots \int_0^{2\pi} \left| \sum_{k=1}^N e^{i\theta_k} \right|^{2p} \prod_{1 \leq i < j \leq N} \left| e^{i\theta_i} - e^{i\theta_j} \right|^{2\gamma} \prod_{k=1}^N d\theta_k \quad (6.73)$$

is known for  $\gamma = 1$  that  $S_N(p, 1)$  gives the number of permutations of  $(1, \dots, k)$  in which the length of the longest increasing subsequence is less than or equal to  $N$ . One has in particular,

$$S_N(p, 1) = p!, \quad 0 \leq p \leq N.$$

It is desirable to know the integrals  $S_N(p, \gamma)$  for a general  $\gamma$ .

The following short proof of Selberg's formula is from [1].

*Anderson's proof of Selberg's Integral.* Anderson's proof depends on Dirichlet's generalization of the beta integral given in the following: For  $\text{Re}(\alpha_j) > 0$ ,

$$\int \dots \int_V p_0^{\alpha_0-1} p_1^{\alpha_1-1} \dots p_n^{\alpha_n-1} dp_0 \dots dp_{n-1} = \frac{\prod_{j=1}^n \Gamma(\alpha_j)}{\Gamma\left(\sum_{j=1}^n \alpha_j\right)}, \quad (6.74)$$

where  $V$  is the set  $p_j \geq 0, \sum_{j=0}^n p_j = 1$ . The formula is used after a change of variables. To see this, first consider Selberg's integral, which may be written as

$$S_n = n! A_n(\alpha, \beta, \gamma) := n! \int_0^1 \int_0^{x_n} \cdots \int_0^{x_2} |\phi(0)|^{\alpha-1} |\phi(1)|^{\beta-1} |\Delta_\phi|^\gamma dx_1 \cdots dx_n, \quad (6.75)$$

where  $0 < x_1 < x_2 < \cdots < x_n < 1$ ,

$$\phi(t) = \prod_{j=1}^n (t - x_j) = t^n - \phi_{n-1} t^{n-1} + \cdots + (-1)^n \phi_0 \quad (6.76)$$

and  $\Delta_\phi$  is the *discriminant* of  $\phi$ , so that

$$|\Delta_\phi| = \left| \prod_{j=1}^n \phi'(x_j) \right| = \left| \prod_{1 \leq i < j \leq n} (x_i - x_j) \right|^2.$$

We now change the variables from  $x_1, \dots, x_n$  to  $\phi_0, \dots, \phi_{n-1}$ , which are the elementary symmetric functions of the  $x_i$ 's. In fact, we have:

$$A_n(\alpha, \beta, \gamma) = \int |\phi(0)|^{\alpha-1} |\phi(1)|^{\beta-1} |\Delta_\phi|^{\gamma-\frac{1}{2}} d\phi_0 d\phi_1 \cdots d\phi_{n-1}, \quad (6.77)$$

where the integration is over all points  $(\phi_0, \phi_1, \dots, \phi_{n-1})$  in which the  $\phi_j$  are elementary symmetric functions of  $x_1, \dots, x_n$  with  $0 < x_1 < \cdots < x_n$ . Indeed, it is sufficient to prove that the Jacobian

$$\det \left( \left[ \frac{\partial \phi_i}{\partial x_j} \right] \right) = \sqrt{|\Delta_\phi|}.$$

Observe that two columns of the Jacobian are equal when  $x_i = x_j$ . Thus  $\prod_{i < j} (x_i - x_j)$  is a factor of the determinant. Moreover, the Jacobian and  $\prod_{i < j} (x_i - x_j)$  are homogeneous and of the same degree. This proves the above Jacobian.

We make a similar change of variables in Eq. (6.74). To accomplish this, set

$$\varphi(t) = \prod_{j=0}^n (t - \zeta_j) \quad (0 \leq \zeta_0 < \zeta_1 < \cdots < \zeta_n < 1)$$

and let

$$\mathcal{D} = \left\{ \prod_{j=1}^n (t - x_j) : \zeta_{j-1} < x_j < \zeta_j; j = 1, \dots, n \right\}. \quad (6.78)$$

Next, we show that: For all  $\phi(t) = t^n - \phi_{n-1} t^{n-1} + \cdots + (-1)^n \phi_0 \in \mathcal{D}$ , the following map

$$(\phi_0, \phi_1, \dots, \phi_{n-1}) \mapsto \left( \frac{\phi(\zeta_0)}{\phi'(\zeta_0)}, \dots, \frac{\phi(\zeta_n)}{\phi'(\zeta_n)} \right) \equiv (p_0, p_1, \dots, p_n) \in \mathbb{R}^{n+1}$$

where  $\varphi'(t)$  denotes the derivative of  $\varphi(t)$ , is a bijection and  $p_j > 0$  with  $\sum_{j=0}^n p_j = 1$ .

Observe that

$$p_j = \frac{\phi(\zeta_j)}{\phi'(\zeta_j)} = \frac{(\zeta_j - x_1)(\zeta_j - x_2) \cdots (\zeta_j - x_n)}{(\zeta_j - \zeta_0) \cdots (\zeta_j - \zeta_{j-1})(\zeta_j - \zeta_{j+1}) \cdots (\zeta_j - \zeta_n)} > 0 \quad (6.79)$$

since the numerator and denominator have exactly  $n - j$  negative factors. Now let  $\varphi_j(t) = \frac{\phi(t)}{t - \zeta_j}$ . By Lagrange's interpolation formula

$$\phi(t) = \sum_{j=0}^n p_j \varphi_j(t) \equiv \sum_{j=0}^n \frac{\varphi_j(t)}{\phi'(\zeta_j)} \phi(\zeta_j). \quad (6.80)$$

One can directly verify this by checking that both sides of the equation are polynomials of degree  $n$  and are equal at  $n + 1$  points  $t = \zeta_j, j = 0, \dots, n$ . Equate the coefficients of  $t^n$  on both sides to get  $\sum_{j=0}^n p_j = 1$ . Now for a given point  $(p_0, p_1, \dots, p_n)$  with  $\sum_{j=1}^n p_j = 1$  and  $p_j > 0, j = 1, \dots, n$ , define  $\phi(t)$  by Eq. (6.80). The expressions

$$\phi(\zeta_j) = p_j \varphi_j(\zeta_j) = p_j (\zeta_j - \zeta_0) \cdots (\zeta_j - \zeta_{j-1})(\zeta_j - \zeta_{j+1}) \cdots (\zeta_j - \zeta_n)$$

and

$$\phi(\zeta_{j+1}) = p_{j+1} \varphi_{j+1}(\zeta_{j+1}) = p_{j+1} (\zeta_{j+1} - \zeta_0) \cdots (\zeta_{j+1} - \zeta_j)(\zeta_{j+1} - \zeta_{j+2}) \cdots (\zeta_{j+1} - \zeta_n)$$

show that  $\phi(\zeta_j)$  and  $\phi(\zeta_{j+1})$  have different signs and  $\phi$  vanishes at some point  $x_{j+1}$  between  $\zeta_j$  and  $\zeta_{j+1}$ . Thus  $\phi \in \mathcal{D}$ . This proves the bijection.

We can now restate Dirichlet's formula Eq. (6.74) as:

$$\int_{\phi(t) \in \mathcal{D}} \prod_{j=0}^n |\phi(\zeta_j)|^{\alpha_j - 1} d\phi_0 \cdots d\phi_{n-1} = \frac{\prod_{j=0}^n |\phi'(\zeta_j)|^{\alpha_j - \frac{1}{2}} \Gamma(\alpha_j)}{\Gamma\left(\sum_{j=1}^n \alpha_j\right)}. \quad (6.81)$$

Indeed,

$$p_j^{\alpha_j - 1} = \left( \frac{\phi(\zeta_j)}{\phi'(\zeta_j)} \right)^{\alpha_j - 1} = \frac{|\phi(\zeta_j)|^{\alpha_j - 1}}{|\phi'(\zeta_j)|^{\alpha_j - 1}},$$

hence

$$\int \cdots \int_V p_0^{\alpha_0 - 1} p_1^{\alpha_1 - 1} \cdots p_n^{\alpha_n - 1} dp_0 \cdots dp_{n-1} \quad (6.82)$$

$$= \frac{1}{\prod_{j=0}^n |\phi'(\zeta_j)|^{\alpha_j - 1}} \int_{\phi(t) \in \mathcal{D}} \prod_{j=0}^n |\phi(\zeta_j)|^{\alpha_j - 1} dp_0 \cdots dp_{n-1}. \quad (6.83)$$

We need to verify that the Jacobian

$$\det \left( \frac{\partial(p_0, \dots, p_{n-1})}{\partial(\phi_0, \dots, \phi_{n-1})} \right) = \prod_{j=0}^n |\phi'(\zeta_j)|^{-\frac{1}{2}},$$

that is,  $dp_0 \cdots dp_{n-1} = \prod_{j=0}^n |\varphi'(\zeta_j)|^{-\frac{1}{2}} d\phi_0 \cdots d\phi_{n-1}$  or  $d\phi_0 \cdots d\phi_{n-1} = \prod_{j=0}^n |\varphi'(\zeta_j)|^{\frac{1}{2}} dp_0 \cdots dp_{n-1}$ .  
 Since

$$p_j = \frac{1}{\varphi'(\zeta_j)} \left( \zeta_j^n - \phi_{n-1} \zeta_j^{n-1} + \cdots + (-1)^n \phi_0 \right),$$

the Jacobian is

$$\left| \det \left( \frac{\partial(p_0, \dots, p_{n-1})}{\partial(\phi_0, \dots, \phi_{n-1})} \right) \right| = \left| \frac{\det \left( \zeta_i^j \right)_{n-1}}{\prod_{j=0}^{n-1} \varphi'(\zeta_j)} \right| = \left| \frac{\det \left( \zeta_i^j \right)_n}{\prod_{j=0}^n \varphi'(\zeta_j)} \right| = \frac{\prod_{j=0}^n |\varphi'(\zeta_j)|^{\frac{1}{2}}}{\prod_{j=0}^n |\varphi'(\zeta_j)|}.$$

The numerator is a Vandermonde determinant and therefore the result follows:

$$\begin{aligned} & \int \cdots \int_V p_0^{\alpha_0-1} p_1^{\alpha_1-1} \cdots p_n^{\alpha_n-1} dp_0 \cdots dp_{n-1} \\ &= \frac{1}{\prod_{j=0}^n |\varphi'(\zeta_j)|^{\alpha_j-\frac{1}{2}}} \int_{\phi(t) \in \mathcal{D}} \prod_{j=0}^n |\phi(\zeta_j)|^{\alpha_j-1} \left( \prod_{j=0}^n |\varphi'(\zeta_j)|^{\frac{1}{2}} dp_0 \cdots dp_{n-1} \right) \\ &= \frac{1}{\prod_{j=0}^n |\varphi'(\zeta_j)|^{\alpha_j-\frac{1}{2}}} \int_{\phi(t) \in \mathcal{D}} \prod_{j=0}^n |\phi(\zeta_j)|^{\alpha_j-1} d\phi_0 \cdots d\phi_{n-1}. \end{aligned}$$

The final step is to obtain the  $(2n-1)$ -dimensional integral. Let  $\phi(t)$  and  $\Phi(t)$  be two polynomials such that

$$\phi(t) = \prod_{i=1}^{n-1} (t - x_i) \quad \text{and} \quad \Phi(t) = \prod_{j=1}^n (t - y_j), \quad (6.84)$$

$$0 < y_1 < x_1 < y_2 < \cdots < x_{n-1} < y_n < 1. \quad (6.85)$$

The resultant of  $\phi$  and  $\Phi$ , denoted  $R(\phi, \Phi)$ , is given by

$$|R(\phi, \Phi)| = \left| \prod_{i \in [n-1], j \in [n]} (x_i - y_j) \right| = \left| \prod_{j=1}^n \phi(y_j) \right| = \left| \prod_{i=1}^{n-1} \Phi(x_i) \right|. \quad (6.86)$$

The absolute value of the discriminant of  $\phi$  can be written as  $|R(\phi, \phi')|$ . That is,

$$|\Delta_\phi| = |\Delta(x)|^2 = \prod_{j=1}^{n-1} \phi'(x_j).$$

The  $(2n-1)$ -dimensional integral is

$$\begin{aligned} & \int_{(\phi, \Phi)} |\Phi(0)|^{\alpha-1} |\Phi(1)|^{\beta-1} |R(\phi, \Phi)|^{\gamma-1} d\phi_0 \cdots d\phi_{n-2} d\Phi_0 \cdots d\Phi_{n-1} \\ &= \int_{(\phi, \Phi)} |\Phi(0)|^{\alpha-1} |\Phi(1)|^{\beta-1} \left| \prod_{j=1}^n \phi(y_j) \right|^{\gamma-1} d\phi_0 \cdots d\phi_{n-2} d\Phi_0 \cdots d\Phi_{n-1}. \end{aligned} \quad (6.87)$$

Here the integration is over all  $\phi$  and  $\Phi$  defined by Eq. (6.84). Then we show that Selberg's integral  $A_n(\alpha, \beta, \gamma)$  satisfies the recurrence relation:

$$A_n(\alpha, \beta, \gamma) = \frac{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma n)}{\Gamma(\alpha + \beta + \gamma(n-1))} A_{n-1}(\alpha + \gamma, \beta + \gamma, \gamma). \quad (6.88)$$

In fact, integrate the  $(2n-1)$ -dimensional integral Eq. (6.87) with respect to  $d\phi_0 \cdots d\phi_{n-2}$  and use  $\Phi(t)$  instead of  $\varphi(t)$  in Eq. (6.81) to get

$$\begin{aligned} & \int_{(\phi, \Phi)} |\Phi(0)|^{\alpha-1} |\Phi(1)|^{\beta-1} \left| \prod_{j=1}^n \phi(y_j) \right|^{\gamma-1} d\phi_0 \cdots d\phi_{n-2} d\Phi_0 \cdots d\Phi_{n-1} \\ &= \int_{\Phi} |\Phi(0)|^{\alpha-1} |\Phi(1)|^{\beta-1} \left( \int_{\phi} \prod_{j=1}^n |\phi(y_j)|^{\gamma-1} d\phi_0 \cdots d\phi_{n-2} \right) d\Phi_0 \cdots d\Phi_{n-1} \\ &= \int_{\Phi} |\Phi(0)|^{\alpha-1} |\Phi(1)|^{\beta-1} \left( \frac{\prod_{j=1}^n |\Phi'(y_j)|^{\gamma-\frac{1}{2}} \Gamma(\gamma)}{\Gamma(\gamma n)} \right) d\Phi_0 \cdots d\Phi_{n-1} \\ &= \frac{\Gamma(\gamma)^n}{\Gamma(\gamma n)} \int_{\Phi} |\Phi(0)|^{\alpha-1} |\Phi(1)|^{\beta-1} \left| \prod_{j=1}^n \Phi'(y_j) \right|^{\gamma-\frac{1}{2}} d\Phi_0 \cdots d\Phi_{n-1} = \frac{\Gamma(\gamma)^n}{\Gamma(\gamma n)} A_n(\alpha, \beta, \gamma). \end{aligned}$$

It remains to compute Eq. (6.87) in another way, set  $\tilde{\phi}(t) = t \prod_{j=1}^n (t - x_j)$ , and

$$\alpha_0 = \alpha, \alpha_j = \gamma (j = 1, \dots, n-1), \alpha_n = \beta; \quad x_0 = 0, x_n = 1$$

so that Eq. (6.87) is equal to

$$\begin{aligned} & \int_{(\phi, \Phi)} |\Phi(0)|^{\alpha-1} |\Phi(1)|^{\beta-1} \left| \prod_{j=1}^{n-1} \Phi(x_j) \right|^{\gamma-1} d\Phi_0 \cdots d\Phi_{n-1} d\phi_0 \cdots d\phi_{n-2} \\ &= \int_{(\phi, \Phi)} \prod_{j=0}^n |\Phi(x_j)|^{\alpha_j-1} d\Phi_0 \cdots d\Phi_{n-1} d\phi_0 \cdots d\phi_{n-2}. \end{aligned} \quad (6.89)$$

Now integrate Eq. (6.89) with respect to  $d\Phi_0 \cdots d\Phi_{n-1}$  and use  $\tilde{\phi}$  instead of  $\varphi(t)$  in Eq. (6.81) to obtain

$$\frac{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)^{n-1}}{\Gamma(\alpha + \beta + \gamma(n-1))} \int_{\phi} \left| \prod_{j=1}^{n-1} \tilde{\phi}'(x_j) \right|^{\gamma-\frac{1}{2}} |\tilde{\phi}'(0)|^{\alpha-\frac{1}{2}} |\tilde{\phi}'(1)|^{\beta-\frac{1}{2}} d\phi_0 \cdots d\phi_{n-2}.$$

Since

$$\begin{aligned} |\tilde{\phi}'(0)| &= \left| \prod_{j=1}^{n-1} x_j \right|, \\ |\tilde{\phi}'(1)| &= \left| \prod_{j=1}^{n-1} (1 - x_j) \right|, \\ \prod_{j=1}^n |\phi'(x_j)| &= \prod_{j=1}^{n-1} |x_j| \prod_{j=1}^{n-1} |1 - x_j| |\Delta_{\phi}|, \end{aligned}$$

the last integral can be written as

$$\begin{aligned} & \frac{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)^{n-1}}{\Gamma(\alpha+\beta+\gamma(n-1))} \int_{\phi} \left( \prod_{j=1}^{n-1} x_j^{\alpha+\gamma-1} (1-x_j)^{\beta+\gamma-1} \right) |\Delta_{\phi}|^{\gamma-\frac{1}{2}} d\phi_0 \cdots d\phi_{n-2} \\ &= \frac{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)^{n-1}}{\Gamma(\alpha+\beta+\gamma(n-1))} A_{n-1}(\alpha, \beta, \gamma). \end{aligned}$$

Equate the two different evaluations of the  $(2n-1)$ -dimensional integral to obtain the result. Finally, Selberg's formula is obtained by iterating Eq. (6.88)  $(n-1)$  times.  $\square$

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